pC

Vol. 1, No. 3 July, 1953

The Journal of Computing Systems

8. A SYSTEM OF MODAL LOGIC.
Jan Łukasiewicz

9. ON COMPLETENESS OF DECISION ELEMENT SETS.
Norman M. Martin

10. SINGLE AXIOMS FOR THE SYSTEMS (C,N), (C,0) AND (A,N) OF THE TWO-VALUED PROPOSITIONAL CALCULUS.
Carew A. Meredith

11. A FORMALIZATION OF SOBOCIŃSKI'S THREE-VALUED IMPLICATIONAL PROPOSITIONAL CALCULUS.
Alan Rose

12. A SINGLE AXIOM OF POSITIVE LOGIC.
Carew A. Meredith

13. NOTE ON A MODAL SYSTEM OF FEYS-VON WRIGHT.
Bolesław Sobociński

14. ANALYTIC MINIMIZATION METHODS I: CONJUNCTIVE FORMS.
W. C. Carter and A. S. Rettig

15. NOTES ON DECISION ELEMENT SYSTEMS USING VARIOUS PRACTICAL TECHNIQUES.
John D. Goodell

Published by THE INSTITUTE OF APPLIED LOGIC
45 W. Water St., St. Paul 1, Minnesota

THE INSTITUTE OF APPLIED LOGIC

The Institute of Applied Logic is a non-profit educational institution incorporated under the laws of the State of Minnesota. The scope of activity and the purposes of the Institute are as follows:

1. To generate a widespread understanding of the science of logic.

2. To develop and demonstrate the application and interpretation of systems of logic in various fields of scientific work.

3. To disseminate knowledge about systems of logic and their application.

4. To develop new systems of logic, new interpretations and new applications.

It is not within the scope of the Institute to study the internal structures of any science or activity beyond the extent necessary to develop suitable interpretations and applications of logical systems. The interest of the Institute in any such application is strictly objective and divorced from any implications involved in the end results. This is intended to define the fact that except for the realm of logic the Institute is not interested in any political, religious, sociological, ethical, moral, industrial, scientific or other systems (or objectives of such systems) per se, but only in the application of the principles of logic in the improved development of such systems.

The program of the Institute for accomplishing its objectives includes the publication of this journal. The journal is intended to link a wide variety of people in the field of computing machinery and in the field of theoretical logic. This requires a broad editorial policy and the publication of articles at a variety of levels. Scholarly papers of principle interest to professional logicians as well as articles pertaining to the development of systems for computing machinery and related material are solicited.

The funds of the Institute are obtained from grants, donations, subscriptions, charges for consulting services and similar sources. Membership contributions, including subscription to the Journal of Computing Systems, are currently established at five dollars annually.

Publication of this Journal is aided by a grant from the Louis W. and Maud Hill Family Foundation to the Institute of Applied Logic for a project pertaining to the study of the internal languages and potential applications of computing machinery.

Manuscripts appropriate to the purposes of the Journal and the Institute are solicited. They should be addressed to the Institute of Applied Logic, 47 West Water St., St. Paul 1, Minnesota. All manuscripts will be given prompt consideration and those accepted will be published as soon as possible. Authors will be given fifty reprints without charge and additional reprints will be supplied upon request at cost.

THE JOURNAL
of
COMPUTING SYSTEMS

• • •

Edited by the Staff of the Institute of Applied Logic under the direction of John D. Goodell.

• • •

Subscription $5.00 per annum
$6.00 outside U. S. and possessions

Published by

THE INSTITUTE OF APPLIED LOGIC
45 West Water St., St. Paul 1, Minn.

NOTE

The term "Computing Systems" is used in the title of this Journal in its broadest sense. It is intended to include logical and mathematical systems as well as structures for machines designed to solve problems that involve computing. Thus the principal purpose of the Journal is to provide a common meeting ground, a channel for communication, in these inter-related fields. Articles will be published covering the interpretation of logical and mathematical methods in the field of computing machinery as well as theoretical technical papers in all three subjects. The editorial policy and the choice of subject matter will be substantially guided by the interest shown and communications from readers will be welcomed.

A SYSTEM OF MODAL LOGIC

JAN ŁUKASIEWICZ

The present essay consists of two parts: the first contains general remarks on systems of modal logic, the second is an exposition of a new modal system.

PART I

1. What is modal logic. A logical system is usually called 'modal logic', if there occur in it modal expressions such as 'possible' or 'necessary'. Instead of this rather vague characterization I shall try to give a precise definition of modal logic according to the tradition initiated by Aristotle.

First I shall explain what I understand by 'basic modal logic'. I am calling thus a system containing the expressions:

'It is possible that p' denoted by 'Δ p', and
'It is necessary that p' denoted by 'Γ p',

if and only if they satisfy the following eight conditions:

I. The implication 'If p, then it is possible that p' is asserted, in symbols:

$$1.1 \quad \vdash Cp\Delta p.$$

'C' means 'if-then', 'p' is a propositional variable, and '$\vdash$' is the sign of assertion.[1]

II. The implication 'If it is possible that p, then p' is rejected, in symbols:

$$1.2 \quad \dashv C\Delta pp.$$

'$\dashv$' is the sign of rejection.[2]

(Received February 20, 1953)

III. The proposition 'It is possible that p' is rejected, in symbols:

$$1.3 \dashv \Delta p.$$

IV. The implication 'If it is necessary that p, then p' is asserted, in symbols:

$$1.4 \vdash C\Gamma pp.$$

V. The implication 'If p, then it is necessary that p' is rejected, in symbols:

$$1.5 \dashv Cp\Gamma p.$$

VI. The proposition 'It is not necessary that p' is rejected, in symbols:

$$1.6 \dashv N\Gamma p.$$

'N' means 'not'.

VII. The equivalence 'It is possible that p - if and only if - it is not necessary that not p' is asserted, in symbols:

$$1.7 \vdash E\Delta pN\Gamma Np.$$

'E' means 'if and only if'. In my symbolic notation the functors are always put before their arguments.

VIII. The equivalence 'It is necessary that p - if and only if - it is not possible that not p' is asserted, in symbols:

$$1.8 \vdash E\Gamma pN\Delta Np.$$

The first condition corresponds to the principle: <u>Ab esse ad posse valet consequentia.</u>

The second condition corresponds to the saying: <u>A posse ad esse non valet consequentia.</u>

The third condition states that not all formulae beginning with Δ are asserted, because otherwise Δ p would be equivalent to the function '<u>verum</u> of p' which is not a modal function.

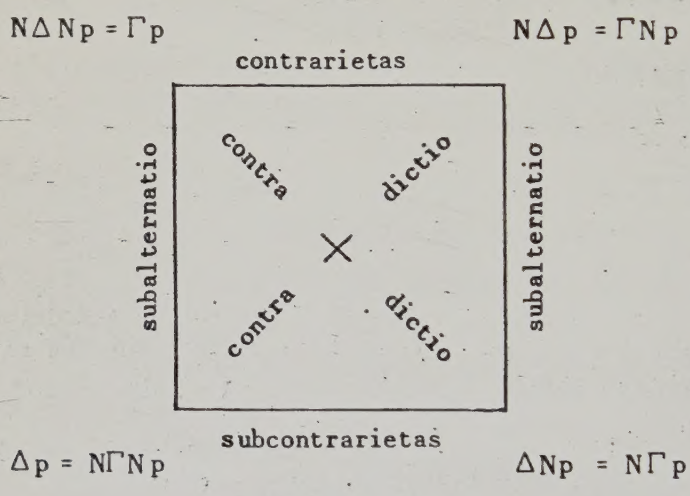
The fourth condition corresponds to the principle: *Ab oportere ad esse valet consequentia.*

The fifth condition corresponds to the saying: *Ab esse ad oportere non valet consequentia.*

The sixth condition states that not all formulae beginning with $N\Gamma$ are asserted, because otherwise Γp would be equivalent to the function '*falsum* of p' which is not a modal function.

The last two conditions are evident relations between possibility and necessity.

The above conditions, except perhaps the third and the sixth, which seem to be unknown to the traditional logicians, are embodied in the following 'square of modalities':



I call a system 'modal logic' if, and only if, it includes the basic modal logic as its part.

I accept throughout the paper that both Δ and Γ are proposition-forming functors of one propositional argument, and that both Δp and Γp are truth-functions, i.e., their truth-values depend only on the truth-values of their arguments. As there exists in the two-valued logic no functor of one argument which would satisfy the formulae 1.1, 1.2, and 1.3, or 1.4, 1.5, and 1.6, it is plain that the basic modal logic, and, consequently, every system of modal logic is a many-valued system.

2. Axiomatization of the basic modal logic. The next step to throw some light upon the modal logic is to axiomatize the basic modal logic on the ground of the classical calculus of propositions. It can be easily seen that of the two modal functors, Δp and Γp, one may be taken as the primitive term, and the other can be defined. Let us take Δ as the primitive term. It would seem that, accepting the first three Δ-formulae as axioms, we could deduce from them the remaining four Γ-formulae. This is, however, not the case: formula 1.7, which contains besides Δ the defined functor Γ, cannot be got in this way, and must be accepted axiomatically. It is not elegant to use defined terms in axioms; we take, therefore, as the fourth axiom, instead of 1.7, the formula $E\Delta p\Delta NNp$, which is equivalent to $E\Delta pN\Gamma Np$. We get thus the following set of axioms:

2.1 $\vdash C p\Delta p$ (= 1.1)

2.2 $\dashv C\Delta pp$ (= 1.2)

2.3 $\dashv \Delta p$ (= 1.3)

2.4 $\vdash E\Delta p\Delta NNp$.

Γp is defined by the equivalence:

Df Γp * 2.5

2.5 $\vdash E\Gamma pN\Delta Np$. (= 1.8)

I accept the usual rules of substitution and detachment for the asserted formulae. The analogous rules for the rejected expressions run thus:

(a) Rule of substitution: If α is rejected, and α is a substitution of β, then β must be rejected.

(b) Rule of detachment: If $C\alpha\beta$ is asserted, and β is rejected, then α must be rejected.

Both rules are evident. Rule (a) is applied below to prove 2.11 and 2.14, rule (b) to prove 2.10 and 2.13.

The deduction.[3]

Auxiliary formulae of the propositional calculus:

T1 $\vdash CEpNqEErNpErq$

T2 ⊢ CEpqCqp

T3 ⊢ CEpNqCCNrqCpr.

Derived formulae of the basic modal logic:

T1 p/ΓNp, q/ΔNNp, r/Δp * C2.5 p/Np-2.6

2.6 ⊢ EEΔpNΓNpEΔpΔNNp

T2 p/EΔpNΓNp, q/EΔpΔNNp * C2.6-C2.4-2.7

2.7 ⊢ EΔpNΓNp (= 1.7)

T3 p/Γp, q/ΔNp, r/p * C2.5-C2.1p/Np-2.8

2.8 ⊢ CΓpp (= 1.4)

T3 p/Δp, q/ΓNp, r/p * C2.7-2.9

2.9 ⊢ CCNpΓNpCΔpp

2.9 * C2.10-2.2

2.10 ⊣ CNpΓNp

2.10 * 2.11 p/Np

2.11 ⊣ CpΓp (= 1.5)

T2 p/Δp, q/NΓNp * C2.7-2.12

2.12 ⊢ CNΓNpΔp

2.12 * C2.13-2.3

2.13 ⊣ NΓNp

2.13 * 2.14 p/Np

2.14 ⊣ NΓp (= 1.6)

Any of the four Δ-axioms is independent of the remaining three. This is easy to prove for the first three axioms. We take for C and N the normal two-valued matrix M1, and show the independence of 2.1, 2.2 and 2.3 by interpreting Δp as Np, p and Vp respectively. I shall explain the last proof. Vp, i.e., '<u>verum</u> of p', has for all truth-values of p the asserted truth-value of 1. If Δp = Vp, then Δp = 1, since Vp = 1. Δp, therefore, is asserted, i.e., axiom 2.3 is not verified. Axiom 2.1 is verified, because CpΔp = Cp1 = 1; similarly, 2.4 is verified, because EΔpΔNNp = E11 = 1. From CΔpp we get for p/2: CΔ22 = C12 = 2, and as 2 is rejected, CΔpp is rejected too. All our axioms are verified by the matrix M1 and the interpretation Δp = Vp, except 2.3 which,

C	1	2	N
*1	1	2	2
2	1	1	1

M1

therefore, is independent of the remaining axioms. The other proofs are of a similar kind.

The proof of independence of the fourth axiom requires a three-valued matrix, M2.[4] M2 verifies the C-N-axioms of the classical calculus of propositions, Δ is defined independently, and Γ is got from N and Δ according to the definition 2.5. 1 is the asserted truth-value, the other truth-values are rejected. It can be easily seen that CpΔp is asserted. CΔpp is rejected, because we have for p/2: CΔ22 = C12 = 2. Similarly, Δp is rejected, because for p/3 we have Δ3 = 3. The axioms 2.1, 2.2 and 2.3 are thus verified, but axiom 2.4 is not verified, because its consequence CΔpΔNNp is rejected for p/2: CΔ2ΔNN2 = C1ΔN1 = C1Δ3 = C13 = 3. Since 2.4, i.e., EΔpΔNNp, is equivalent to EΔpNΓNp, this last formula is not verified too. Formula CpΓp is also not verified, since it must be asserted according to the matrix, whereas it should be rejected. It is clear, therefore, that axiom 2.4 is indispensable for the axiomatization of the basic modal logic.

C	1	2	3	N	Δ	Γ
*1	1	2	3	3	1	1
2	1	1	1	1	1	3
3	1	1	1	1	3	3

M 2

We get a corresponding set of axioms of the basic modal logic, if we take Γ as the primitive term and accept as axioms the following four Γ-formulae:

2.15 ⊢ CΓpp (= 1.4)
2.16 ⊣ CpΓp (= 1.5)
2.17 ⊣ NΓp (= 1.6)
2.18 ⊢ EΓpΓNNp.

Δ is introduced by the definition:

Df Δp * 2.19
2.19 ⊢ EΔpNΓNp. (= 1.7)

Formula EΓpΓNNp is equivalent to EΓpNΔNp. This can be proved by T1 in the same way as the equivalence of EΔpΔNNp and EΔpNΓNp, by interchanging in 2.6 the Δ's and Γ's. 2.18 is independent of the remaining axioms. The proof is given by the matrix M3 with two asserted truth-values, which verifies the C-N-axioms

C	1	2	3	N	Γ
*1	1	2	3	3	1
*2	1	1	3	3	3
3	1	2	1	1	3

M3

of the classical calculus of propositions and the axioms 2.15-2.17, but does not verify 2.18, as the consequence of 2.18, CΓNNpΓp is rejected for p/2: CΓNN2Γ2 = CΓN33 = CΓ13 = C13 = 3.

3. Aristotle's theorems of the propositional modal logic. It is a pity that the formulae of the modal square never were correctly axiomatized on the basis of the classical calculus of propositions, and that even the problem of such an axiomatization never was clearly seen.[5] Nobody, therefore, could observe that two odd formulae are hidden in the square, viz.:

EΔpΔNNp and EΓpΓNNp

which are indispensable for a correct axiomatization. These formulae throw a light on the modal logic just because of their similar shape: they suggest the idea that there must be a general principle independent of the modal square from which they may be deduced. There are still other reasons to suppose that the basic modal logic is not complete and requires the addition of some new principles. So, for instance, we believe that if a conjunction is possible, each of its factors should be possible, in symbols:

3.1 ⊢ CΔKpqΔp

3.2 ⊢ CΔKpqΔq;

and if a conjunction is necessary, each of its factors should be necessary, in symbols:

3.3 ⊢ CΓKpqΓp

3.4 ⊢ CΓKpqΓq.

None of these formulae can be deduced from the modal square, i.e., from the basic modal system.

It is strange enough that the only two theorems of the modal logic which Aristotle expressly states with propositional variables can be interpreted so as to give us the general principle we are looking for. Referring to his syllogisms, Aristotle writes in the Prior Analytics: 'If one should denote the premises by α, and the conclusion by β, it would not only result that if α is necessary, then β is necessary, but also if α is possible, then β is possible.'[6] There are two different ways of interpreting these

theorems as formulae of modal logic, although it is highly improbable that Aristotle was aware of their difference. Let us explain these two interpretations.

All the Aristotelian syllogisms are implications of the form $C\alpha\beta$ where α is the conjunction of the two premises and β the conclusion. E.g., 'If all a is b and all b is c, then all a is c', in symbols:

$$C\underbrace{KAabAbc}_{\alpha}\underbrace{Aac}_{\beta}.$$ [7]

According to the above quotation, we get two modal theorems taking $C\alpha\beta$ as the antecedent, and $C\Delta\alpha\Delta\beta$ or $C\Gamma\alpha\Gamma\beta$ as the consequent, in symbols:

3.5 $\vdash CC\alpha\beta C\Delta\alpha\Delta\beta$ and

3.6 $\vdash CC\alpha\beta C\Gamma\alpha\Gamma\beta$.

The letters α and β stand here for the premises and the conclusion of an Aristotelian syllogism. We may treat these theorems as special examples of general principles which we get by replacing the Greek letters by propositional variables:

3.7 $\vdash CCpqC\Delta p\Delta q$ and

3.8 $\vdash CCpqC\Gamma p\Gamma q$.

This is the first interpretation. The principle for Δ seems to be confirmed by Aristotle himself in a second passage which reads quite generally: 'It has been proved that if (if α is, β is), then (if α is possible, β is possible).'[8] Formula 3.7 is accepted as Aristotelian by A. Becker and I.M. Bocheński.[9]

We get a second interpretation if we draw attention to the fact that according to Aristotle the connection between the premises α of a syllogism and its conclusion β is necessary. This gives us the special theorems:

3.9 $\vdash C\Gamma C\alpha\beta C\Delta\alpha\Delta\beta$ and

3.10 $\vdash C\Gamma C\alpha\beta C\Gamma\alpha\Gamma\beta$,

which we may extend into the principles:

3.11 $\vdash C\Gamma CpqC\Delta p\Delta q$ and

3.12 $\vdash C\Gamma CpqC\Gamma p\Gamma q$.

The principle 3.11 for Δ seems to be corroborated by Aristotle himself, as we read at the beginning of the same chapter where the other modal theorems occur: 'First it has to be said that if (if α is, β must be), then (if α is possible, β must be possible too).'[10] The second 'must' evidently refers to the necessary connection between the antecedent and the consequent, but the first 'must' seems to state a necessary connection between α and β in the antecedent. There is no reference to a syllogism.

The formulae got by the first interpretation are stronger than those got by the second, as it is shown by the following deduction:

T4 ⊢ CCpqCCqrCpr

1.4 ⊢ CΓpp

T4 p/ΓCpq, q/Cpq, r/CΔpΔq * C1.4p/Cpq-C3.7-3.11

3.11 ⊢ CΓCpqCΔpΔq

T4 p/ΓCpq, q/Cpq, r/CΓpΓq * C1.4p/Cpq-C3.8-3.12

3.12 ⊢ CΓCpqCΓpΓq.

We see that 3.11 follows from 3.7 and 3.12 from 3.8 by means of 1.4 and the principle of the hypothetical syllogism T4. The converse deduction is not valid. This can be proved by the matrix M4 which results for C and N from the multiplication of M1 by itself, verifies the C-N-axioms, the basic modal system, and the formulae 3.11 and 3.12, but does not verify 3.7 and 3.8, as we have for p/4, q/2: CC42CΔ4Δ2 = C1C32 = C12 = 2, and for p/3, q/1: CC31CΓ3Γ1 = C1C32 = C12 = 2.

C	1	2	3	4	N	Γ	Δ
*1	1	2	3	4	4	2	1
2	1	1	3	3	3	2	2
3	1	2	1	2	2	3	3
4	1	1	1	1	1	4	3

M4

4. Possible extensions of the basic modal logic. All the four newly introduced principles, the stronger 3.7 and 3.8 as well as the weaker 3.11 and 3.12, are independent of the basic modal system on the ground of the classical calculus of propositions. It suffices to prove this for the weaker principles, because if these are shown to be independent, the stronger must be independent too. The proof is given by the eight-valued matrix M5 which re-

sults for C and N from the multiplication of the matrix M1 by the matrix M4. M5 verifies the C-N-axioms and the basic modal logic, but does not verify 3.11 and 3.12, as we get for p/5, q/6:

$$C\Gamma C56C\Delta 5\Delta 6 = C\Gamma 2C16 = C26 = 5$$

and for p/3,q/4:

$$C\Gamma C34C\Gamma 3\Gamma 4 = C\Gamma 2C38 = C26 = 5$$

C	1	2	3	4	5	6	7	8	N	Γ	Δ
*1	1	2	3	4	5	6	7	8	8	1	1
2	1	1	3	3	5	5	7	7	7	2	1
3	1	2	1	2	5	6	5	6	6	3	1
4	1	1	1	1	5	5	5	5	5	8	1
5	1	2	3	4	1	2	3	4	4	8	1
6	1	1	3	3	1	1	3	3	3	8	6
7	1	2	1	2	1	2	1	2	2	8	7
8	1	1	1	1	1	1	1	1	1	8	8

M5

The new principles are not only independent of, but also consistent with the basic modal logic on the ground of the C-N-system. The proof of consistency is given by the matrix M6 which is identical

C	1	2	3	4	N	Γ	Δ
*1	1	2	3	4	4	2	1
2	1	1	3	3	3	2	1
3	1	2	1	2	2	4	3
4	1	1	1	1	1	4	3

M6

with M4 for C and N, but different for Γ and Δ. M6 verifies the C-N-axioms, the basic modal logic, and all the four principles got by interpretation of Aristotle's modal theorems. Any such principle, when added to the basic modal logic, will expand this system into a fuller one.

The formula $CEpqC\phi p\phi q$ is called in logic 'the principle of extensionality for ϕ'. In a wider sense we may also thus call the formulae $CCpqC\phi p\phi q$ and $CCpqC\phi q\phi p$, because we get from them by $CEpqCpq$ or $CEpqCqp$ and the hypothetical syllogism the principle $CEpqC\phi p\phi q$. For instance, the principle of transposition $CCpqCNqNp$ is in a wider sense a principle of extensionality for N, because we get from it the formula $CEpqCNpNq$. The principles $CCpqC\Delta p\Delta q$ and $CCpqC\Gamma p\Gamma q$ are in a wider sense principles of extensionality for Δ and Γ.

These two principles are equivalent to each other on the ground of the C-N-system and the basic modal logic. Starting from 3.7 (CCpqCΔ pΔ q) we get 3.8 by means of the formulae:

T5 $\vdash$ CCNqNpCrsCCpqCNsNr

T6 $\vdash$ CEsqCEtrCCpCqrCpCst

2.5 $\vdash$ EΓpNΔNp

T5 r/Δ Nq, s/Δ Np * C3.7p/Nq, q/Np-4.1

4.1 $\vdash$ CCpqCNΔ NpNΔ Nq

T6 s/Γ p, q/NΔ Np, t/Γ q, r/NΔ Nq, p/Cpq * C2.5-C2.5p/q-C4.1-3.8

3.8 $\vdash$ CCpqCΓ pΓ q.

The converse deduction from 3.8 to 3.7 can be performed in the same way by interchanging Δ and Γ.

If we add 3.7 to the Δ-axioms of the basic modal logic we get by the laws of the C-N-system the formula EΔ pΔ NNp:

T7 $\vdash$ CpNNp

T8 $\vdash$ CNNpp

T9 $\vdash$ CCpqCCqpEpq

3.7 q/NNp * CT7-4.2

4.2 $\vdash$ CΔ pΔ NNp

3.7 p/NNp, q/p * CT8-4.3

4.3 $\vdash$ CΔ NNpΔ p

T9 p/Δ p, q/Δ NNp * C4.2-C4.3-2.4

2.4 $\vdash$ EΔ pΔ NNp

In the same way we can prove EΓ pΓ NNp starting from 3.8.

Owing to the stronger interpretation of the Aristotelian theorems we have found in the principle of extensionality for modal functors the general law from which the formulae EΔ pΔ NNp and EΓpΓ NNp of the modal square can be deduced.

The extended modal system which arises by the addition of CCpqCΔ pΔ q to the basic modal logic and is expounded in the second part of this article is the simplest complete modal logic with an adequate four-valued matrix. It is, in my opinion, both logically and philosophically of the highest importance. Never-

theless, it is wholly unknown. All the existing systems of modal logic, as far as I see, extend the basic modal logic by weaker principles, assuming either such formulae as CΓ CpqCΓ pΓ q or CΓ CpqCΔ pΔ q which correspond to the weaker interpretation of the Aristotelian theorems or rules of extensionality instead of principles. The principles of extensionality for modal functors are not accepted. In Von Wright's system, for instance, formula CCpqCNΔ qNΔ p, which is equivalent to CCpqCΔ pΔ q, is expressly disproved.[11] All these modal systems are possible extensions of the basic modal logic and may have their own merits; perhaps we shall be able to decide some day which of them is the best.

PART II

5. Axioms of the Ł-modal system. The modal system expounded in this part presupposes the classical calculus of propositions, accepts Δ as the only primitive modal term, and is built up on the basic modal logic with the addition of only one new modal principle, viz., the strict principle of extensionality for Δ.

The general principle of extensionality, taken sensu stricto, has the form:

5.1 ⊢ CEpqCδ pδ q

where δ is a variable functor.[12] This principle I extend to the modal functor Δ getting thus the formula:

5.1 δ/Δ' * 5.2
5.2 ⊢ CEpqCΔ pΔ q.

Formula 5.2 seems to be intuitively evident. We say that if p and q are equivalent to each other, then 'If p is true, q is true', and 'If p is false, q is false'; so we may also say that under the same condition 'If p is possible, q is possible'. Von Wright accepts in his system the rule of extensionality:

5.3 ⊢ $E\alpha\beta \rightarrow$ ⊢ $E\Delta\alpha\Delta\beta$,

in words: 'α if and only if β; therefore, α is possible, if and

only if β is possible.'[13] The arrow is the sign of 'therefore'. Rule 5.3 follows from the formula 5.2.

The general principle of extensionality 5.1 must be accepted in the Ł-modal system, as it is valid for all functors of one argument of the classical calculus of propositions, and is admitted for the modal functor Δ. This leads to a simplification of the axiom-set of the system. From 5.1 by means of the formulae of the classical C-N-system:

T10 ⊢ CCpqCpCrq

T11 ⊢ CCpCqrCpCqCsr

T12 ⊢ CCprCCqrCApqr

T13 ⊢ AEpqENpq

we get the following consequences:

T10 p/ENpq, q/Cδ Np δ q, r/δ p * C5.1p/Np - 5.4

5.4 ⊢ CENpqC δ pC δ Np δ q

T11 p/Epq, q/δ p, r/δ q, s/δ Np * C5.1 - 5.5

5.5 ⊢ CEpqC δ pC δ Np δ q

T12 p/Epq, r/C δ pC δ Np δ q, q/ENpq * C5.5 -C5.4 -CT13 -5.6

5.6 ⊢ C δ pC δ Np δ q.

It was shown by C. A. Meredith - in an unpublished paper - that formula 5.6 may be taken as the sole axiom of the classical C-N-δ-p-calculus, i.e., the classical C-N-calculus of propositions extended by the addition of variable functors. I accept, therefore, as the first axiom of the Ł-modal system the formula:

1 ⊢ C δ pC δ Np δ q.

From this axiom I derive by substitution and detachment the three axioms of the C-N-system:[14]

22 ⊢ CCpqCCqrCpr

20 ⊢ CCNppp

10 ⊢ CpCNpq,

the principle of extensionality:

73 ⊢ CEpqC δ p δ q,

and the fourth axiom of the basic modal logic, 2.4:

89 $\vdash E\Delta p\Delta NNp$.

The other three axioms of the basic modal logic are independent of axiom 1, and must be taken axiomatically:

2 $\vdash Cp\Delta p$

3 $\dashv C\Delta pp$

4 $\dashv \Delta p$,

so that our modal system is based on four axioms: 1, 2, 3 and 4. As rules of inference I accept the rules of substitution and detachment for asserted and rejected formulae.

The proofs of independence are the same as in the basic modal logic. The independence of axiom 1 is proved by the matrix M2, because we have for δ/Δ' and p/2,q/3: $C\Delta 2C\Delta N2\Delta 3 = C1C\Delta 13 = C1C13 = C13 = 3$.

The Aristotelian principle:

3.7 $\vdash CCpqC\Delta p\Delta q$

which is a principle of extensionality in a wider sense, is stronger than the strict principle of extensionality 5.2. Nevertheless, it is deducible in our system by means of the law

30 $\vdash C\delta CpqC\delta p\delta q$,

a consequence of the axiom 1. We get from this law by substitution δ/Δ' the formula:

77 $\vdash C\Delta CpqC\Delta p\Delta q$,

and as $CCpq\Delta Cpq$ is true according to our axiom 2, we get

78 $\vdash CCpqC\Delta p\Delta q$

by the help of the syllogism. We may say, therefore, that our system arises from the basic modal logic by the addition of an Aristotelian principle.

6. Matrix of the Ł-modal system. We get an adequate matrix of the Ł-modal system by 'multiplying' the matrices M7 and M8, both identical with the adequate matrix M1 of the two-valued cal-

culus, but with different figures as elements in order to avoid misunderstandings. The figures 5 and 7 marked by an asterisk are the selected elements, i.e., the asserted values, 6 and 8 are rejected.

C	5	6	N
*5	5	6	6
6	5	5	5

M7

C	7	8	N
*7	7	8	8
8	7	7	7

M8

The process of multiplication can be described as follows:

First, we form ordered pairs of elements of both matrices by combining an element of M7 with an element of M8; we get thus four combinations:

*(5,7), (5,8), (6,7), (6,8).

These combinations are the elements of the new matrix. The selected element is (5,7), as 5 and 7 are the selected elements of the original matrices.

Secondly, we determine the truth-values of the functions C, N and Δ by means of the following equalities (a,b,d represent the elements of M7, x,y,z the elements of M8):

6.1 C(a,x)(b,y) = (Cab,Cxy),

6.2 N(a,x) = (Na,Nx),

6.3 Δ(a,x) = (a,Cxx).

Substituting for a and b the values 5 and 6, for x and y the values 7 and 8, and evaluating the functions on the right according to the matrices M7 and M8, e.g.: C(6,7)(6,8) = (C66,C78) = (5,8), we get from these equalities the following matrix M9:

C	(5,7)	(5,8)	(6,7)	(6,8)	N	Δ
*(5,7)	(5,7)	(5,8)	(6,7)	(6,8)	(6,8)	(5,7)
(5,8)	(5,7)	(5,7)	(6,7)	(6,7)	(6,7)	(5,7)
(6,7)	(5,7)	(5,8)	(5,7)	(5,8)	(5,8)	(6,7)
(6,8)	(5,7)	(5,7)	(5,7)	(5,7)	(5,7)	(6,7)

M9

M9 is an adequate matrix of the system, i.e., it verifies all

its formulae and no other formulae besides. This can be seen by the following consideration.

First, M9 verifies the axioms of the C-N-calculus, CCpqCCqrCpr, CCNppp, and CpCNpq. Putting in these axioms for the variables arbitrary elements of the matrix, (a,x), (b,y), and (d,z), we get by means of 6.1 and 6.2 the following equalities:

6.4 CC(a,x)(b,y)CC(b,y)(d,z)C(a,x)(d,z) = C(Cab,Cxy)C(Cbd,Cyz)(Cad,Cxz) = C(Cab,Cxy)(CCbdCad,CCyzCxz) = (CCabCCbdCad,CCxyCCyzCxz) = (5,7).

6.5 CCN(a,x)(a,x)(a,x) = CC(Na,Nx)(a,x)(a,x) = C(CNaa,CNxx)(a,x)(CCNaaa,CCNxxx) = (5,7).

6.6 C(a,x)CN(a,x)(b,y) = C(a,x)C(Na,Nx)(b,y) = C(a,x)(CNab,CNxy)(CaCNab,CxCNxy) = (5,7).

The final result in all cases is the selected element (5,7) of M9, as CCabCCbdCad, for instance, always gives 5 according to M7, and CCxyCCyzCxz always gives 7 according to M8.

Secondly, M9 verifies the axioms 2, 3, and 4. We have by 6.1 and 6.3:

6.7 C(a,x)Δ(a,x) = C(a,x)(a,Cxx) = (Caa,CxCxx) = (5,7).

6.8 CΔ(a,x)(a,x) = C(a,Cxx)(a,x) = (Caa,CCxxx),

which gives (5,8) for x/8, a rejected element of M9.

6.9 Δ(a,Cxx),

which gives (6,7) for a/6, again a rejected element of M9.

Thirdly, in order to prove that CδpCδNpδq is verified by M9, it suffices to show that the principle of extensionality CEpqCδpδq or CCpqCCqpCδpδq is verified by M9 for all functors of one argument definable by M9. There are 16 such functors, as we can combine in 16 ways the four functions of M7, V(verum), S(assertion), N(negation), and F(falsum) with the analogous four functions of M8, e.g., (Va,Nx), (Sa,Fx), and so on. All these functions, however, are reducible to C-N-formulae, because Va = Caa, Sa = a, Fa = NCaa, and likewise Vx = Cxx, Sx = x and Fx = NCxx.

By substituting, therefore, the new functors for δ we get C-N-formulae, and all such formulae are verified by M9. Take, for example, the principle of extensionality for Δ:

6.10 $CC(a,x)(b,y)CC(b,y)(a,x)C\Delta(a,x)\Delta(b,y) =$
$C(Cab,Cxy)C(Cba,Cyx)C(a,Cxx)(b,Cyy) =$
$C(Cab,Cxy)C(Cba,Cyx)(Cab,CCxxCyy) =$
$C(Cab,Cxy)(CCbaCab,CCyxCCxxCyy) =$
$(CCabCCbaCab,CCxyCCyxCCxxCyy) = (5.7)$.

It follows from this consideration that all the formulae of our modal logic based on the axioms 1-4 are verified by the matrix M9. It also follows that no other formulae besides can be verified by M9; this results from the fact that the classical C-N-propositional calculus, to which all the formulae of our modal logic are matrically reducible, is 'saturated', i.e., any formula must be either asserted on the ground of its asserted axioms, or rejected on the ground of the axiom of rejection $\dashv$ p, which easily follows by substitution from our axiom 3 or 4. M9, therefore, is an adequate matrix of the Ł-modal logic.

Let us now write, for the sake of abbreviation, 1 for (5,7), 2 for (5,8), 3 for (6,7), and 4 for (6,8); we get from M9 the matrix M6 which is the adequate matrix of our modal system in its simplest form.

C	1	2	3	4	N	Δ
*1	1	2	3	4	4	1
2	1	1	3	3	3	1
3	1	2	1	2	2	3
4	1	1	1	1	1	3

M6

7. <u>The Twin Possibilities.</u> A curious logical fact is connected with the definition of Δ, which, as far as I know, has not yet been observed. The formulae with Δ are obviously a product of formulae verified by S (assertion) and V (verum). $Cp\Delta p$ is asserted because it is asserted for $\Delta = S$ and $\Delta = V$. $C\Delta pp$ and Δp are rejected because the first formula is rejected for $\Delta = V$, and the second for $\Delta = S$. Now we can obtain a product of S and V by multiplying S by V, getting thus the function $\Delta(a,x) = (Sa,Vx) = (a,Cxx)$, or by multiplying V by S getting $(Va,Sx) = (Caa,x)$. Let us denote this latter function by an inverted Δ:

7.1 $\nabla(a,x) = (Caa,x)$.

From 6.1, 6.2, and 7.1 there results the following matrix:

C	(5,7)	(5,8)	(6,7)	(6,8)	N	∇
*(5,7)	(5,7)	(5,8)	(6,7)	(6,8)	(6,8)	(5,7)
(5,8)	(5,7)	(5,7)	(6,7)	(6,7)	(6,7)	(5,8)
(6,7)	(5,7)	(5,8)	(5,7)	(5,8)	(5,8)	(5,7)
(6,8)	(5,7)	(5,7)	(5,7)	(5,7)	(5,7)	(5,8)

M10

I shall now abbreviate this matrix by replacing the pairs of elements by single figures. As it does not matter which figures we choose, let us write 1 for (5,7), 2 for (6,7), 3 for (5,8), and 4 for (6,8). We get the matrix M6a which is identical with M6,

C	1	3	2	4	N	∇
*1	1	3	2	4	4	1
3	1	1	2	2	2	3
2	1	3	1	3	3	1
4	1	1	1	1	1	3

M6a

as we can easily see by interchanging the middle lines and columns. Consequently, M10 is identical with M9, and the functor ∇ defined in this way is identical with the functor Δ.

We encounter here a logical paradox: although Δ and ∇ can be defined by the same matrix, they are not identical. Let us apply to ∇ in M10 the abbreviation of M9: 1 for (5,7), 2 for (5,8), 3 for (6,7), and 4 for (6,8): we get for C,N, and Δ the matrix M6, and for ∇ the matrix M11, which is different from Δ. Δ and ∇ are undistinguishable when they occur separately, but their difference appears at once when they occur in the same formula. They are like twins who cannot be distinguished when met separately, but are instantly recognized as two when seen together. Take, for instance, the formulae $\Delta\Delta p$, $\nabla\nabla p$, $\Delta\nabla p$, and $\nabla\Delta p$. $\Delta\Delta p$ is equivalent to Δp which is rejected, and likewise $\nabla\nabla p$ is equivalent to ∇p which is rejected too. But $\nabla\Delta p$ and $\Delta\nabla p$ must be asserted according to M6 and M11. We cannot, therefore, replace in the two last formulae Δ by ∇, or vice versa, although

	∇
1	1
2	2
3	1
4	2

M11

both functors can be defined by the same matrix.

In the two-valued logic the asserted value, denoted by 1, is called 'truth', the rejected value, denoted usually by 0, 'falsity'. When I had discovered in 1920 a three-valued system of logic, I called the third value, which I denoted by ½, 'possibility'.[15] Later on, after having found my n-valued modal systems, I thought that only two of them may be of philosophical importance, viz., the 3-valued and the $\aleph_0$-valued system. For we can assume, I argued, that either possibility has no degrees at all getting thus the 3-valued system, or that it has infinitely many degrees, as in the theory of probabilities, and then we have the $\aleph_0$-valued system.[15] This opinion, as I see it today, was wrong. The Ł-modal logic is a 4-valued system with two values, 2 and 3, denoting possibility, but, nevertheless, both values represent one and the same possibility in two different shapes. The values 2 and 3 are playing in the system exactly the same role which can be seen by the following table of the 16 functions of one argument:

(A)	(B)	(D)	(G)		(H)
(a,x)	(5,7)(5,8)(6,7)(6,8)	1234	p	p	1234
(a,Nx)	(5,8)(5,7)(6,8)(6,7)	2143	CΔNpNCΔpp	C∇pNC∇NpNp	3412
(a,Vx)	(5,7)(5,7)(6,7)(6,7)	1133	Δp	C∇pp	1212
(a,Fx)	(5,8)(5,8)(6,8)(6,8)	2244	NΔNp	NC∇NpNp	3434
(Na,x)	(6,7)(6,8)(5,7)(5,8)	3412	CΔpNCΔNpNp	C∇NpNC∇pp	2143
(Na,Nx)	(6,8)(6,7)(5,8)(5,7)	4321	Np	Np	4321
(Na,Vx)	(6,7)(6,7)(5,7)(5,7)	3311	ΔNp	C∇NpNp	2121
(Na,Fx)	(6,8)(6,8)(5,8)(5,8)	4422	NΔp	NC∇pp	4343
(Va,x)	(5,7)(5,8)(5,7)(5,8)	1212	CΔpp	∇p	1133
(Va,Nx)	(5,8)(5,7)(5,8)(5,7)	2121	CΔNpNp	∇Np	3311
(Va,Vx)	(5,7)(5,7)(5,7)(5,7)	1111	Cpp	Cpp	1111
(Va,Fx)	(5,8)(5,8)(5,8)(5,8)	2222	NΔNCpp	∇NCpp	3333
(Fa,x)	(6,7)(6,8)(6,7)(6,8)	3434	NCΔNpNp	N∇Np	2244
(Fa,Nx)	(6,8)(6,7)(6,8)(6,7)	4343	NCΔpp	N∇p	4422
(Fa,Vx)	(6,7)(6,7)(6,7)(6,7)	3333	ΔNCpp	N∇NCpp	2222
(Fa,Fx)	(6,8)(6,8)(6,8)(6,8)	4444	NCpp	NCpp	4444

The first column (A) represents the 16 functions, the second

(B) contains their matrices for a/5,x/7 - a/5,x/8 - a/6,x/7 - a/6,x/8, the third (D) is the translation of these matrices according to the equalities: (5,7) = 1, (5,8) = 2, (6,7) = 3, (6,8) = 4, the fourth column (G) gives the formulae corresponding to (D) for p = 1,2,3,4 according to the matrices M6 and M11 (e.g., C∆NpNC∆pp in the second line has the value 2 for p/1, 1 for p/2, 4 for p/3 and 3 for p/4), and the last column (H) is the translation of the matrices (B) according to the equalities: (5,7) = 1, (5,8) = 3, (6,7) = 2 and (6,8) = 4, ordered for p = 1,2,3,4. We get the figures of the last column from the column (D) by writing 3 for 2 and 2 for 3, and then by interchanging the middle figures, e.g., from 2143 we get first 3142, and then 3412.

It can be easily seen that the figures of the last column are matrices of the corresponding formulae with ∇ and, if we define ∆p as C∇pp, also of the formulae with ∆. Assuming 1133 as the matrix of ∇p we get, e.g., for C∇pp the matrix 1212, because C∇11 = C11 = 1, C∇22 = C12 = 2, C∇33 = C33 = 1, C∇44 = C34 = 2, and defining ∆p as C∇pp we have for C∆pp the matrix 1133, because C∆11 = C11 = 1, C∆22 = C22 = 1, C∆33 = C13 = 3, and C∆44 = C24 = 3. Now, the formulae with ∇ and their corresponding matrices of column (H) are identical with the formulae with ∆ and their corresponding matrices of column (D), as ∇ and ∆ are identical functors. We see, therefore, that we get the same formulae by interchanging 2 and 3, and that these twin values of possibility play in the system the same role.

It also follows from the table that although all those 16 functions are reducible to the C-N-system by the matrices (B), the functions corresponding to the abbreviated matrices (D) and (H) cannot be defined in this way. The modal functor ∆ or its twin ∇ is necessary and sufficient to represent them together with C and N.

8. Some Formulae of the Ł-Modal Logic. The classical system of the propositional calculus extended by the addition of variable functors is not yet universally known. This system inspired by Leśniewski's 'Protothetic' was modified by myself by introduction of the rule of δ-substitution. Owing to this rule, we get easy and elegant proofs. By means of them I deduce in the Appendix from axiom 1 first the three axioms of the C-N-calculus, viz.:

10	⊢ CpCNpq	the principle of Duns Scotus,
20	⊢ CCNppp	the principle of Clavius,
22	⊢ CCpqCCqrCpr	the principle of the syllogism,

and then some other formulae without and with δ needed for the modal logic. Among the latter formulae the most important are the following ones: The principle of extensionality,

27 $\vdash CCpqCCqpC\delta p\delta q$ or 73 $\vdash CEpqC\delta p\delta q$,

the principles of δ-distribution with respect to C and A,

30 $\vdash C\delta CpqC\delta p\delta q$ and 71 $\vdash C\delta ApqA\delta p\delta q$,

and a principle of conjunction,

60 $\vdash C\delta pC\delta q\delta Kpq$.

From these auxiliary theses a considerable number of Δ- and Γ-formulae are derived in the Appendix. Here an account of the most important of them.

(a) The basic modal logic is a part of our system. Three formulae of this system:

2 $\vdash Cp\Delta p$, 3 $\dashv C\Delta pp$ and 4 $\dashv \Delta p$,

are taken as axioms, and the remaining five:

129 $\vdash C\Gamma pp$ 156 $\dashv Cp\Gamma p$, 157 $\dashv N\Gamma p$,

124 $\vdash E\Gamma pN\Delta Np$ and 128 $\vdash E\Delta pN\Gamma Np$,

are proved as consequences.

(b) There are only four modal functors of one argument in the system, viz., Δp (= $N\Gamma Np$), $N\Delta p$ (= ΓNp), ΔNp (= $N\Gamma p$), and $N\Delta Np$ (= Γp). This easily results from two principles of reduction for Δ:

94 $\vdash E\Delta\Delta p\Delta p$ and 98 $\vdash E\Delta N\Delta p\Delta Np$.

The corresponding principles for Γ run:

136 $\vdash E\Gamma\Gamma p\Gamma p$ and 141 $\vdash E\Gamma N\Gamma p\Gamma Np$.

It should be stressed that according to these principles a problematic proposition is equivalent to a problematic one, and an

apodeictic proposition to an apodeictic one.

(c) There are three principles of Δ-distribution for C,K, and A (a fourth one, for E, is easily deducible from the first):

84 $\vdash E\Delta CpqC\Delta p\Delta q$,

109 $\vdash E\Delta KpqK\Delta p\Delta q$,

114 $\vdash E\Delta ApqA\Delta p\Delta q$;

and two principles of Γ-distribution for K and A:

149 $\vdash E\Gamma KpqK\Gamma p\Gamma q$ and 154 $\vdash E\Gamma ApqA\Gamma p\Gamma q$.

The principle of Γ-distribution for C is not valid, because the formula:

162 $\dashv CC\Gamma p\Gamma q\Gamma Cpq$

is rejected.

(d) No apodeictic proposition, i.e., no proposition beginning with Γ or with $N\Delta$, can be asserted in the system. This follows from the formulae:

134 $\vdash C\Gamma qCp\Gamma p$ and 101 $\vdash CN\Delta pC\Delta pp$.

Both formulae are asserted, but their consequents $Cp\Gamma p$ and $C\Delta pp$ are rejected; their antecedents, therefore, Γq and $N\Delta q$, must be rejected too. Now, from the rejected formulae Γq or $N\Delta q$ nothing can be got by substitution, not even the formula:

160 $\dashv \Gamma Cpp$;

on the contrary, from 160 there results by substitution $\dashv \Gamma q$. On the other side, it is obvious that in the asserted formulae 134 and 101 any expression whatever may be put for q, and all formulae got in this way from Γq and $N\Delta q$ must be rejected. In order to express the fact that any proposition beginning with Γ or $N\Delta$ should be rejected, I employ Greek variables, calling them 'interpretation-variables' in opposition to the 'substitution-variables' denoted by Latin letters. We have, therefore:

$\dashv \Gamma\alpha$ and $\dashv N\Delta\alpha$,

where α may be any formula, i.e., any significant expression of the system.

(e) On the contrary, our system contains many asserted problematic propositions. It follows from axiom 2 that if α is an asserted proposition, then the proposition 'It is possible that α' must be asserted too. We have, for instance:

102 $\vdash \Delta C p p.$

There are besides Δ-formulae whose argument is rejected, e.g.:

92 $\vdash \Delta C \Delta p p.$

This problematic proposition is asserted, although its argument $C\Delta pp$ is rejected. Another interesting example is given by the

163 $\vdash \Delta \nabla p.$

It is most difficult to express this formula in the ordinary language. Both Δ and ∇ may be rendered by the phrase 'it is possible that', as both have exactly the same meaning. Nevertheless, they are different, and we cannot say 'It is possible that it is possible that p', because this may have the meaning $\Delta\Delta p$, and $\Delta\Delta p$ cannot be asserted being equivalent to Δp.

The list of modal formulae given in the Appendix should be completed by modal formulae with δ. So, for instance, it can be proved that the following formulae are asserted:

$\Delta C\delta\Delta p\delta p$, $\Delta C\delta p\delta\Delta p$, $\Delta C\delta p\delta\Gamma p$, $\Delta C\delta\Gamma p\delta p$, $\Delta E\delta p\delta\Delta p$, $\Delta E\delta p\delta\Gamma p$.

All such formulae are put off to a further investigation.

9. <u>Some Controversial Problems.</u> Hitherto, the best known systems of modal logic are originated by C. I. Lewis. It is difficult to compare my own modal logic with them, as they are based on the so-called 'strict implication' which is stronger than the 'material implication' employed by myself. I shall compare, therefore, my system with the systems of G. H. Von Wright which are also based on the material implication and are equivalent, according to its author, to some systems of Lewis.

There are three modal systems presented by Von Wright in axiomatic form, and called by him M, M′, and M″.[16] All are based on the classical calculus of propositions and on two additional

rules of transformation:

9.1. The 'Rule of Extensionality': 'If $f_1 \longleftrightarrow f_2$ is provable, then $Mf_1 \longleftrightarrow Mf_2$ is also provable'. That means in my symbolism:

$$\vdash E\alpha\beta \rightarrow \vdash E\Delta\alpha\Delta\beta \quad (\text{"M"} = \text{"}\Delta\text{"}).$$

9.2 The 'Rule of Tautology'. 'If f is provable, then Nf is provable.' That means:

$$\vdash \alpha \rightarrow \vdash \Gamma\alpha \quad (\text{"N"} = \text{"}\Gamma\text{"}).$$

System M is established on two modal axioms:

9.3 $a \rightarrow Ma$ the 'Axiom of Possibility'

which corresponds to our asserted formula 2 $Cp\Delta p$, and

9.4 $M(a \vee b) \longleftrightarrow Ma \vee Mb$ the 'Axiom of Distribution'

which corresponds to our asserted formula 114 $E\Delta ApqA\Delta p\Delta q$.

System M′ arises from M by addition of the 'First Axiom of Reduction':

9.5 $MMa \rightarrow Ma$

which corresponds to our asserted formula 93 $C\Delta\Delta p\Delta p$, and M″ is got by addition of the 'Second Axiom of Reduction':

9.6 $M{\sim}M\alpha \rightarrow {\sim}M\alpha$

which corresponds to our rejected formula 121 $C\Delta N\Delta pN\Delta p$.

This last axiom gives, together with its converse formula $CN\Delta p\Delta N\Delta p$ (which results from $Cp\Delta p$ by the substitution $p/N\Delta p$), the equivalence $E\Delta N\Delta pN\Delta p$. Here a problematic proposition $\Delta N\Delta p$ appears to be equivalent to an apodeictic proposition $N\Delta p$, which is against our logical intuitions. The author himself seems to be doubtful about this axiom. I think that it should be rejected, and the system M″ is not acceptable.

The systems M and M′ are clearly incomplete, as 9.6, not being inconsistent with them, does not follow from them. The consistency of 9.6 with the rest of the system results from the one fact, among others, that for the interpretation $Ma = a$ all the axioms and rules remain valid. It may be added that in this case the system ceases

to be a modal logic. As Von Wright does not accept rejection, I do not know how he can disprove the formula Ma → a (= $C\Delta pp$) on the ground of his axiomatic system.

All his other axioms and rules, i.e., 9.1, 9.3, 9.4, and 9.5 are valid in my Ł-modal logic, except rule 9.2. This controversial rule, first stated by Aristotle, but not exactly enough, was the cause of many philosophical and theological discussions.[17] After a long, but - in my opinion - unconvincing argumentation Von Wright says: '. . . the proposition that a tautology is necessary and a contradiction impossible are truths of logic. This certainly agrees with our logical intuitions.'[18] I am not certain that it does agree. I think, roughly speaking, that true propositions are simply true without being necessary, and false propositions are simply false without being impossible. This certainly does not hurt our logical intuitions, and may settle many controversies.

It may be asked, however: Why should we introduce necessity and impossibility into logic if true apodeictic propositions do not exist? I reply to this objection that we are primarily interested in problematic propositions of the form $\Delta\alpha$ and $\Delta N\alpha$ which may be true and useful, although their arguments are rejected, and introducing problematic propositions we cannot omit their negations, i.e., apodeictic propositions, as both are inextricably connected with each other.

The second controversial problem concerns the formula 108 $\vdash CK\Delta p\Delta q\Delta Kpq$. In some of his systems Lewis accepts the formula 106 $\vdash C\Delta KpqK\Delta p\Delta q$, but rejects its converse 108 by the following argument: 'If it is possible that p and q be both true, then p is possible and q is possible. This implication is not reversible. For example: it is possible that the reader will see this at once. It is also possible that he will not see it at once. But it is not possible that he will both see it at once and not see it at once.'[19] As this argument is stated in words, and not in symbols, it is equally applicable to the strict as to the material implication. But its evidence is illusive. What is meant by 'the reader'? If an individual reader, say R, is meant, then R either will see this at once, or R will not see this at once. In the first case, the proposition, 'It is possible that R will see this at once', is true; but how can it be proved that R will possibly not see this at once? In the second case, the proposition, 'It is possible that R will not see this at once', is true; but how can it be proved that R will possibly see this at once? The two premises of the formula 108 are not both provable, and the formula cannot be refuted in this way.

Take another example. Let _n_ be a positive integer. I contend that the following implication is true for all values of _n_:

'If it is possible that *n* is even, and it is possible that *n* is not even, then it is possible that *n* is even and *n* is not even.' If *n* = 4, it is true that *n* is possibly even, but it is not true that *n* is possibly not even; if *n* is 5, it is true that *n* is possibly not even, but it is not true that *n* is possibly even. The both premises are never true together, and the formula cannot be refuted.

If again by 'the reader' some reader is meant, then the propositions, 'It is possible that some reader will see this at once', and 'It is possible that some reader will not see this at once', may be both true, but in this case the consequent, 'It is possible that some reader will see this at once and some reader will not see this at once', is obviously also true. It is, of course, not the same reader who will possibly see this and possibly not see this at once. I cannot find an example which would refute formula 108; on the contrary, all seem to support its correctness.

I am fully aware that other systems of modal logic are possible based on different concepts of necessity and possibility. I firmly believe that we shall never be able to decide which of them is true. Systems of logic are instruments of thought, and the more useful a logical system is, the more valuable it is. I hope that the Ł-modal system expounded above will be a useful instrument, and deserves a further investigation and development.

APPENDIX

Examples of δ-substitution and δ-definitions.

(a) Proof-line of formula 16:

1 δ/CpC'p * C15q/p-C10q/p-16

Write CpC'p instead of the δ's filling up the gaps marked by the apostrophe with the arguments of δ. You get thus from

CδpCδNpδq

the formula

CCpCppCCpCNppCpCqp,

from which there follows by two detachments CpCqp.

(b) Proof-line of formula 10:

1 δ /′ * 10

Cancel simply the δ's in 1.

(c) Proof-line of formula 13:

1 δ /C″ ,p/CpCNpNq,q/p * C11-C12-13

Perform first the substitutions for the propositional variables

CδCpCNpNqCδNCpCNpNqδp,

and write instead of the δ's their arguments α in form of Cαα:

$$C\underbrace{CCpCNpNqCpCNpNq}\,C\underbrace{CNCpCNpNqNCpCNpNq}\underbrace{Cpp}$$

which is C11-C12-13.

(d) All δ-definitions have the form CδPδQ, where P and Q are the <u>definiens</u> and the <u>definiendum</u>. P may be replaced everywhere by Q. Take as an example the proof-line of formula 55:

5 δ /C′p * C52-55

By replacing NCpNq by Kpq according to example (a) we get from 52 the formula 55.

The numbers in brackets after a formula F refer to formulae to which F is applied. For instance, 3 is applied to 118.

Axioms

1 $\vdash$ CδpCδNpδq (10,11,13,15,16,18,19,22,23,25,27,29,30, 38, 44,48,50,60,70)

2 $\vdash$ CpΔp (74,75,78,90,94,102)

3 $\dashv$ CΔpp (118)

4 $\dashv$ Δp (115)

Definitions

Df Kpq * 5

5 ⊢ CδNCpNqδKpq (55,56,57)

Df Apq * 6

6 ⊢ CδCNpqδApq (66,67,68,69,71)

Df Epq * 7

7 ⊢ CδKCpqCqpδEpq (72,73)

Df Γp * 8

8 ⊢ CδNΔNpδΓp (123,124,126,129,131,132)

Df ∇p * 9

9 ⊢ CδCΔppδ∇p (163)

Consequences of Axiom 1

1 δ/' * 10

10 ⊢CpCNpq (11,12,14,16,17,28,35,66,139)

1 δ/CpCNp',p/q,q/Nq * C10-11

11 ⊢CCpCNpNqCpCNpNq (13)

10 p/CpCNpNq,q/NCpCNpNq * C10q/Nq-12

12 ⊢CNCpCNpNqNCpCNpNq (13)

1 δ/C″, p/CpCNpNq,q/p * C11-C12-13

13 ⊢Cpp (14,15,18,19,20,23,32,34,45,102,123,135,155)

10 p/Cpp * C13-14

14 ⊢CNCppq (15)

1 δ/C'Cpp,p/Cpp * C13p/Cpp-C14q/Cpp-15

15 ⊢CqCpp (16,23,26,30,38,47)

1 δ/CpC'p * C15q/p-C10q/p-16

16 ⊢CpCqp (17,21,22,24,26,49,53,59,67,79,91,133)

16 p/CqCNqr,q/CNqCqr * C10p/q,q/r-17

17 ⊢CCNqCqrCqCNqr (18)

1 δ/CC'CqrCqC'r,p/q,q/p * C13p/CqCqr-C17-18

18 ⊢CCpCqrCqCpr (20,32,33,34,35,51,63,100,134)

1 δ/C'p * C13-19

19 ⊢CCNppCqp (20,21)

18 p/CNpp,q/Cpp,r/p * C19q/Cpp-C13-20

20 ⊢CCNppp (24,28,36)

16 p/CCNrrCpr,q/CpNr * C19p/r,q/p-21

21 ⊢CCpNrCCNrrCpr (22)

1 δ/CCp'CC'rCpr, p/r * C16p/Cpr,q/Crr-C21-22

22 ⊢CCpqCCqrCpr (33,36,50,62,64,75,78,81,85,86,90, 99,101,133,139,142)

1 δ/CCpNpC'Np * C13p/CpNp-C15q/CpNp,p/Np-23

23 ⊢CCpNpCqNp (25,34)

16 p/CCNppp * C20-24

24 ⊢CqCCNppp (25)

1 δ/CCpNpCCNpp' * C24q/CpNp-C23q/CNpp-25

25 ⊢CCpNpCCNppq (27)

16 p/CCppCδpδp,q/Cpp * C15q/Cpp,p/δp-26

26 ⊢CCppCCppCδpδp (27)

1 δ/CCp'CC'pCδpδ' * C26-C25q/CδpδNp-27

27 ⊢CCpqCCqpCδpδq (28,40,58,65)

27 q/CNpp * C10q/p-C20-28

28 ⊢CδpδCNpp (29,45,47,49)

28 δ/Cδ'CδNqδq,p/q * C1p/q-29

29 ⊢CδCNqqCδNqδq (30)

1 δ/CδC'qCδ'δq,p/q,q/p * C15q/δCqq,p/δq -C29-30

30 ⊢CδCpqCδpδq (31,77,130)

30 δ/Cp',p/q,q/r * 31

31 ⊢CCpCqrCCpqCpr (63,135)

18 p/Cpq,q/p,r/q * C13p/Cpq-32

32 ⊢CpCCpqq (103,155)

18 p/Cpq,q/Cqr,r/Cpr * C22-33

33 ⊢CCqrCCpqCpr (37,143)

18 p/CpNp,q/Cpp,r/Np * C23q/Cpp-C13-34

34 ⊢CCpNpNp (54)

18 q/Np,r/q * C10-35

35 ⊢ CNpCpq (36,52,80,90,99)

22 p/NNp,q/CNpp,r/p * C35p/Np,q/p-C20-36

36 ⊢ CNNpp (37,39,40,88,117)

33 q/NNp,r/p,p/q * C36-37

37 ⊢ CCqNNpCqp (38)

1 δ/CCNpN'C'p * C15q/CNpNp-C37q/Np-38

38 ⊢ CCNpNqCqp (39,41,42,50,101,158)

38 p/NNp,q/p * C36p/Np-39

39 ⊢ CpNNp (40,87)

27 p/NNp,q/p * C36-C39-40

40 ⊢ CδNNpδp (41,42,43,76,83,128)

40 δ/CCNp'CNqp,p/q * C38q/Nq-41

41 ⊢ CCNpqCNqp (43,52,53,126)

40 δ/CC'NqCqNp * C38p/Np-42

42 ⊢ CCpNqCqNp (54,121,125)

40 δ/CC'qCNqNp * C41p/Np-43

43 ⊢ CCpqCNqNp (74,85,86,96,137)

1 δ/C'q,q/r * 44

44 ⊢ CCpqCCNpqCrq (46,48)

28 δ/Cδ'δp * C13p/δp-45

45 ⊢ CδCNppδp (46)

45 δ/CCpqCCNpq',p/q * C44r/Nq-46

46 ⊢ CCpqCCNpqq (92,140)

28 δ/CCprCCprC'r * C15q/Cpr,p/Cpr-47

47 ⊢ CCprCCprCCNppr (48)

1 δ/CCprCC'rCCNp'r * C47-C44q/r,r/CNpNp
-48

48 ⊢ CCprCCqrCCNpqr (68,82)

28 δ/Cδ'CNδpδp * C16p/δp,q/Nδp-49

49 ⊢ CδCNppCNδpδp (69)

22 p/Nδp,q/CNδNpNδq,r/CδqδNp *
C1δ/Nδ'-C38p/δNp,q/δq-50

50 ⊢ CNδpCδqδNp (51,140)

18 p/Nδp,q/δq,r/δNp * C50-51

51 ⊢ CδqCNδpδNp (70)

41 q/CpNq * C35q/Nq-52

52 ⊢ CNCpNqp (55)

41 p/q, q/CpNq * C16p/Nq, q/p-53

53 ⊢ CNCpNqq (56)

42 p/CpNp, q/p * C34-54

54 ⊢ CpNCpNp (57)

5 δ/C'p * C52-55

55 ⊢ CKpqp (58, 62, 104, 144)

5 δ/C'q * C53-56

56 ⊢ CKpqq (63, 105, 145)

5 δ/Cp', q/p * C54-57

57 ⊢ CpKpp (58)

27 q/Kpp * C57-C55q/p-58

58 ⊢ CδpδKpp (59)

16 p/CδpδKpp, q/δp * C58-59

59 ⊢ CδpCδpδKpp (60)

1 δ/CδpCδ'δKp' * C59-C1q/KpNp-60

60 ⊢ CδpCδqδKpq (61, 72, 107, 147)

60 δ/Cp', p/q, q/r * 61

61 ⊢ CCpqCCprCpKqr (106, 146)

22 p/Kpq, q/p * C55-62

62 ⊢ CCprCKpqr (64)

18 p/CKpqCqr, q/CKpqq, r/CKpqr * C31p/Kpq -C56-63

63 ⊢ CCKpqCqrCKpqr (64)

22 p/CpCqr, q/CKpqCqr, r/CKpqr * C62r/Cqr -C63-64

64 ⊢ CCpCqrCKpqr (65, 108, 148)

64 p/Cpq, q/Cqp, r/Cδpδq * C27-65

65 ⊢ CKCpqCqpCδpδq (73)

6 δ/Cp' * Є10-66

66 ⊢ CpApq (110, 150)

6 δ/Cq' * C16p/q, q/Np-67

67 ⊢CqApq (111,151)

6 δ/CCprCCqrC'r * C48-68

68 ⊢CCprCCqrCApqr (112,152)

6 δ/Cδ'CNδpδp, q/p * C49-69

69 ⊢CδAppCNδpδp (70)

1 δ/CδAp'CNδpδ' * C69-C51q/ApNp-70

70 ⊢CδApqCNδpδq (71)

6 δ/CδApq', p/δp, q/δq * C70-71

71 ⊢CδApqAδpδq (113,153)

7 δ/CCpqCCqp' * C60δ/', p/Cpq, q/Cqp-72

72 ⊢CCpqCCqpEpq (84,89,94,98,109,114,124,127,136, 141,149,154)

7 δ/C'Cδpδq * C65-73

73 ⊢CEpqCδpδq

Δ-Formulae

43 q/Δp * C2-74

74 ⊢CNΔpNp (75,76,95)

22 p/NΔp, q/Np, r/ΔNp * C74-C2p/Np-75

75 ⊢CNΔpΔNp (81)

40 δ/CNΔNp' * C74p/Np-76

76 ⊢CNΔNpp (129)

30 δ/Δ' * 77

77 ⊢CΔCpqCΔpΔq (78,84,93,97)

22 p/Cpq, q/ΔCpq, r/CΔpΔq * C2p/Cpq-C77-78

78 ⊢CCpqCΔpΔq (79,80,85,87,88,91,95,96,104,105, 110,111)

78 p/q, q/Cpq * C16 p/q, q/p-79

79 ⊢CΔqΔCpq (82)

78 p/Np, q/Cpq * C35-80

80 ⊢CΔNpΔCpq (81)

22 p/NΔp, q/ΔNp, r/ΔCpq * C75-C80-81

81 ⊢CNΔpΔCpq (82)

48 p/NΔp, r/ΔCpq, q/Δq * C81-C79-82

82 ⊢CCNNΔpΔqΔCpq (83)

40 δ/CC'ΔqΔCpq, p/Δp * C82-83

83 ⊢CCΔpΔqΔCpq (84)

72 p/ΔCpq, q/CΔpΔq * C77-C83-84

84 ⊢EΔCpqCΔpΔq

22 p/Cpq, q/CΔpΔq, r/CNΔqNΔp * C78-C43p/Δp, q/Δq-85

85 ⊢CCpqCNΔqNΔp (86, 99)

22 p/Cpq, q/CNqNp, r/CNΔNpNΔNq * C43-C85p/Nq, q/Np-86

86 ⊢CCpqCNΔNpNΔNq (131)

78 q/NNp * C39-87

87 ⊢CΔpΔNNp (89)

78 p/NNp, q/p * C36-88

88 ⊢CΔNNpΔp (89, 115)

72 p/Δp, q/ΔNNp * C87-C88-89

89 ⊢EΔpΔNNp

22 p/NΔp, q/CΔpp, r/ΔCΔpp * C35p/Δp, q/p -C2p/CΔpp-90

90 ⊢CNΔpΔCΔpp (92)

78 q/CΔpp * C16 q/Δp-91

91 ⊢CΔpΔCΔpp (92)

46 p/Δp, q/ΔCΔpp * C91-C90-92

92 ⊢ΔCΔpp (93, 96, 163)

77 p/Δp, q/p * C92-93

93 ⊢CΔΔpΔp (94)

72 p/ΔΔp, q/Δp * C93-C2p/Δp-94

94 ⊢EΔΔpΔp

78 p/NΔp, q/Np * C74-95

95 ⊢CΔNΔpΔNp (98)

78 p/CΔpp, q/CNpNΔp * C43p/Δp, q/p-C92-96

96 ⊢ΔCNpNΔp (97)

77 p/Np, q/NΔp * C96-97

97 ⊢CΔNpΔNΔp (98)

72 p/ΔNΔp, q/ΔNp * C95-C97-98

98 ⊢EΔNΔpΔNp

22 p/Np, q/Cpq, r/CNΔqNΔp * C35-C85-99

99 ⊢CNpCNΔqNΔp (100)

18 p/Np, q/NΔq, r/NΔp * C99-100

100 ⊢CNΔqCNpNΔp (101)

22 p/NΔq, q/CNpNΔp, r/CΔpp * C100-C38q/Δp -101

101 ⊢CNΔqCΔpp (118)

2 p/Cpp * C13-102

102 ⊢ΔCpp (103)

32 p/ΔCpp * C102-103

103 ⊢CCΔCppqq (119)

78 p/Kpq, q/p * C55-104

104 ⊢CΔKpqΔp (106)

78 p/Kpq * C56-105

105 ⊢CΔKpqΔq (106)

61 p/ΔKpq, q/Δp, r/Δq * C104-C105-106

106 ⊢CΔKpqKΔpΔq (109)

60 δ/Δ' * 107

107 ⊢CΔpCΔqΔKpq (108)

64 p/Δp, q/Δq, r/ΔKpq * C107-108

108 ⊢CKΔpΔqΔKpq (109)

72 p/ΔKpq, q/KΔpΔq * C106-C108-109

109 ⊢EΔKpqKΔpΔq

78 q/Apq * C66-110

110 ⊢CΔpΔApq (112)

78 p/q, q/Apq * C67-111

111 ⊢CΔqΔApq (112)

68 p/Δp, r/ΔApq, q/Δq * C110-C111-112

112 ⊢CAΔpΔqΔApq (114)

71 δ/Δ' * 113

113 ⊢CΔApqAΔpΔq (114)

72 p/ΔApq, q/AΔpΔq * C113-C112-114

114 ⊢ EΔApqAΔpΔq

* * *

88 * C115-4

115 ⊣ ΔNNp (116)

115 * 116p/Np

116 ⊣ ΔNp (117)

36 p/ΔNp * C117-116

117 ⊣ NNΔNp (157)

101 q/NΔCpp * C118-3

118 ⊣ NΔNΔCpp (119)

103 q/NΔNΔCpp * C119-118

119 ⊣ CΔCppNΔNΔCpp (120)

119 * 120p/Cpp

120 ⊣ CΔpNΔNΔp (121,122)

42 p/ΔNΔp,q/Δp * C121-120

121 ⊣ CΔNΔpNΔp

120 * 122p/Δp

122 ⊣ CpNΔNp (156)

Γ-Formulae

8 δ/Cδ'δNΔNp * C13p/δNΔNp-123

123 ⊢CδΓpδNΔNp (124,125,156,157)

72 p/Γp,q/NΔNp * C123δ/'-C8δ/'-124

124 ⊢EΓpNΔNp

42 p/ΓNp,q/ΔNNp * C123δ/',p/Np-125

125 ⊢ CΔ NNpNΓ Np (127)

41 p/ΔNNp,q/ΓNp * C8δ/',p/Np-126

126 ⊢CNΓNpΔNNp (127)

72 p/ΔNNp,q/NΓNp * C125-C126-127

127 ⊢EΔNNpNΓNp (128)

40 δ/EΔ'NΓNp * C127-128

128 ⊢EΔpNΓNp

8 δ/C'p * C76-129

129 ⊢CΓpp (136,137,142)

30 δ/Γ' * 130

130 ⊢CΓCpqCΓpΓq

8 δ/CCpqC'NΔNq * C86-131

131 ⊢CCpqCΓpNΔNq (132)

8 δ/CCpqCΓp',p/q * C131-132

132 ⊢CCpqCΓpΓq (133,138,139,144,145,150,151)

22 p/q,q/Cpq,r/CΓpΓq * C16p/q,q/p-C132
-133

133 ⊢CqCΓpΓq (134,135)

18 q/Γq,r/Γp * C133q/p,p/q-134

134 ⊢CΓqCpΓp (160)

31 p/Γp,q/Γp,r/ΓΓp * C133q/Γp-C13p/Γp
-135

135 ⊢CΓpΓΓp (136)

72 p/ΓΓp,q/Γp * C129p/Γp-C135-136

136 ⊢EΓΓpΓp

43 p/Γp, q/p * C129-137

137 ⊢CNpNΓp (138)

132 p/Np,q/NΓp * C137-138

138 ⊢CΓNpΓNΓp (141)

22 p/Γp,q/CNΓpNp,r/CΓNΓpΓNp *
C10p/Γp,q/Np-C132p/NΓp, q/Np-139

139 ⊢CΓpCΓNΓpΓNp (140)

46 p/Γp,q/CΓNΓpΓNp * C139-C50δ/Γ',q/NΓp
-140

140 ⊢CΓNΓpΓNp (141,142)

72 p/ΓNΓp,q/ΓNp * C140-C138-141

141 ⊢EΓNΓpΓNp

22 p/ΓNΓp,q/ΓNp,r/Np * C140-C129p/Np-142

142 ⊢CΓNΓpNp (143)

33 q/ΓNΓp,r/Np,p/NΓp * C142-143

143 ⊢CCNΓpΓNΓpCNΓpNp (159)

132 p/Kpq,q/p * C55-144

144 ⊢CΓKpqΓp (146)

132 p/Kpq * C56-145

145 ⊢CΓKpqΓq (146)

61 p/ΓKpq, q/Γp, r/Γq * C144-C145-146

146 ⊢CΓKpqKΓpΓq (149)

60 δ/Γ' * 147

147 ⊢CΓpCΓqΓKpq (148)

64 p/Γp, q/Γq, r/ΓKpq * C147-148

148 ⊢CKΓpΓqΓKpq (149)

72 p/ΓKpq, q/KΓpΓq * C146-C148-149

149 ⊢EΓKpqKΓpΓq

132 q/Apq * C66-150

150 ⊢CΓpΓApq (152)

132 p/q, q/Apq * C67-151

151 ⊢CΓqΓApq (152)

68 p/Γp, r/ΓApq, q/Γq * C150-C151-152

152 ⊢CAΓpΓqΓApq (154)

71 δ/Γ' * 153

153 ⊢CΓApqAΓpΓq (154)

72 p/ΓApq, q/AΓpΓq * C153-C152-154

154 ⊢EΓApqAΓpΓq

32 p/CΓpΓp, q/ΓCpp * C13p/Γp-155

155 ⊢CCCΓpΓpΓCppΓCpp (161)

* * *

123 δ/Cp' * C156-122

156 ⊣ CpΓp (158, 160)

123 δ/N' * C157-117

157 ⊣ NΓp

38 p/Γp, q/p * C158-156

158 ⊣ CNΓpNp (159)

143 * C159-158

159 ⊣ CNΓpΓNΓp

134 q/Cpp * C160-156

160 ⊣ ΓCpp (161)

155 * C161-160

161 ⊣ CCΓpΓpΓCpp (162)

161 * 162q/p

162 ⊣ CCΓpΓqΓCpq

* * *

9 δ/Δ' * C92-163

163 ⊢ Δ∇p

NOTES

1. The idea of assertion and its sign '⊢' were introduced into logic by Frege in 1879, and afterwards accepted by the authors of the *Principia Mathematica*. In my previous papers I always omitted this sign, but here I am bringing it in because, besides assertion, I introduce rejection.

2. The idea of rejection was introduced into logic by myself in 1951. See J. Łukasiewicz, *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*, Oxford 1951, p. 109. I denote rejection by an inverted sign of assertion following a suggestion of Ivo Thomas.

3. For an explanation of the symbolism used in the deduction see my book on Aristotle's Syllogistic, pp.81 and 96.

4. I owe this matrix to C. A. Meredith.

5. The only logician, so far as I know, who saw this problem and tried to solve it, was I. M. Bocheński. His solution, however, is not correct, since the equivalence EΓpNΔNp is not deducible from his axioms. See I. M. Bocheński, *La logique de Théophraste*, Collectanea Friburgensia, Fasc. 32, Fribourg en Suisse 1947, p.92 Sect. 31.

6. An. pr., A 15, 34^a22: εἴ τις θείη τὸ μὲν Α τὰς προτάσεις, τὸ δὲ Β τὸ συμπέρασμα, συμβαίνοι ἂν οὐ μόνον ἀναγκαίου τοῦ Α ὄντος ἅμα καὶ τὸ Β εἶναι ἀναγκαῖον, ἀλλὰ καὶ δυνατοῦ δυνατόν.

7. See my book on Aristotle's Syllogistic, p. 20f.

8. Ibid. 34^a29: δέδεικται ὅτι εἰ τοῦ Α ὄντος τὸ Β ἔστι, καὶ δυνατοῦ ὄντος τοῦ Α ἔσται τὸ Β δυνατόν.

9. See A. Becker, *Die Aristotelische Theorie der Möglichkeitsschlüsse*, Berlin 1933, p.42 note, and I.M. Bocheński, *Ancient Formal Logic*, Studies in Logic and the Foundations of Mathematics, Amsterdam 1951, p.71. Both authors refer to the passage 34^a5 quoted in note[10], which rather supports the second interpretation.

10. Ibid. 34ᵃ5: πρῶτον δὲ λεκτέον ὅτι εἰ τοῦ A ὄντος ἀνάγκη τὸ B εἶναι, καὶ δυνατοῦ ὄντος τοῦ A δυνατὸν ἔσται καὶ τὸ B ἐξ ἀνάγκης.

11. See G. H. Von Wright, An Essay in Modal Logic, Studies in Logic and the Foundations of Mathematics, Amsterdam 1951, p.22/3.

12. A short explanation of δ-substitution and δ-definition is given in the Appendix. For a detailed explanation see J. Łukasiewicz, On Variable Functors of Propositional Arguments, Proceedings of the Royal Irish Academy, Vol. 54 A 2, Dublin 1951.

13. See Von Wright, l. c., p. 85. The symbolism and wording is mine.

14. Formulae marked with numbers without the decimal point are given in the Appendix.

15. See J. Łukasiewicz, Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls, Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie, Vol. 23, Cl.3, 1930, pp. 65 ff. and 72.

16. See l.c. p. 84/5.

17. In an essay on Aristotle's Modal Logic, which will be published elsewhere, I am expounding at length the Aristotelian opinions on this subject.

18. See l. c. p. 14/5.

19. See C. I. Lewis and C. H. Langford, Symbolic Logic, New York and London (1932), p. 167.

DUBLIN, IRELAND

ON COMPLETENESS OF DECISION ELEMENT SETS

NORMAN M. MARTIN

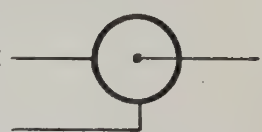
In 'The Foundations of Computing Machinery', Goodell[1] indicated a partial test for completeness of a set of decision elements of radix 2 and order 2. It should be noted that this partial test gives a necessary but not a sufficient condition. It is possible to give a necessary and sufficient condition by reference to results obtained independently by Wernick[2] and by Post[3]. This criterion can be expressed as follows: let us consider 4 sets of decision elements. We shall call them standard sets:

Standard Set I: All decision elements of radix 2 and order 2 which have output 1 if both inputs are 1.

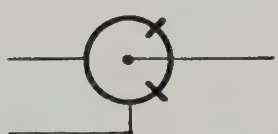
Standard Set II: All decision elements of radix 2, order 2, which have output 0 if both inputs are 0.

Standard Set III: All decision elements of radix 2, order 2, which have output 1 in 0, 2 or 4 of the possible states.

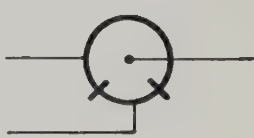
Standard Set IV: The trivial d.e.'s F $\equiv$



B $\equiv$



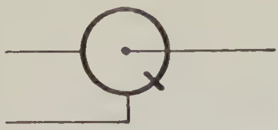
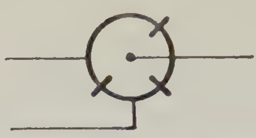
G $\equiv$



the clock d.e. V $\equiv$

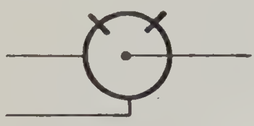
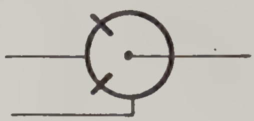
and

(Received July 18, 1953)

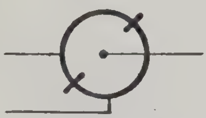
K =  A = 

Theorem: A set of decision elements is complete if and only if it is not a subset of a standard set.

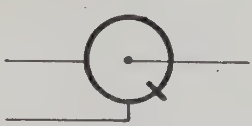
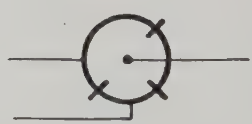
To show necessity one can easily show by induction that 'N' is not realizable in terms of decision elements of I exclusively or in terms of decision elements of II exclusively. To show necessity for III, we consider a minimal formula A in Polish notation which expresses the synthesis of a decision element not in III. If the first sign of A expresses G or B, one of the arguments must express a decision element not in III and A is not minimal. If the first sign of A expresses

J =  or P = 

the formula resulting from the substitution of its negation for it will also express a decision element not in III, reducing to the previous case. Since F and V give the same output regardless of input, they could not be the first sign of A. Since the

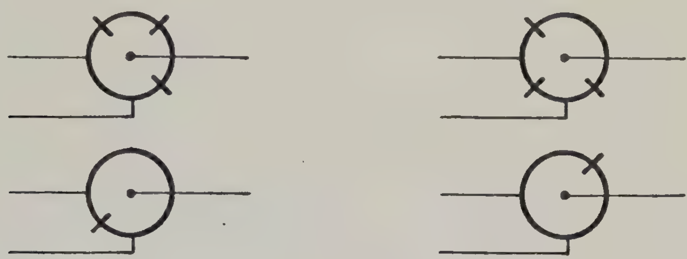
table for R =  corresponds to addition modulo 2,

it could not be the first sign of A unless A were not minimal. Since Epq = RNpq, this case reduces to the last. As to IV, by a similar proof, we see that none of the trivial decision elements could be the first sign of a minimal formula expressing a decision element not in IV and the properties of

K =  and A = 

also clearly show that they could not be the first sign of such a minimal formula (the relevant properties are Kp1 = p, Kp0 = 0, Kpq = Kqp, Ap1 = 1, Ap0 = p, Apq = Aqp). The sufficiency can be shown by showing that N and A or N and K, both known to be complete

sets, can be synthesized by all other combinations (see Table I). It is to be noted that whereas only G and B are trivially realizable in the theory of truth functions, in computer logic F and V are also, the first by disconnection and the second by connection to the clock pulse. Accordingly:



operate as universal elements in practice despite the fact that they are not so in the theory of truth functions[4].

NOTES

1. John D. Goodell, 'The Foundations of Computing Machinery', Journal of Computing Systems, Vol.1, No. 1 (June, 1952), pp. 5-6.

2. William L. Wernick, 'Complete Sets of Logical Functions', Transactions Am. Math. Soc., Vol. 51 (1941) pp. 117-132.

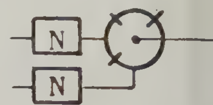
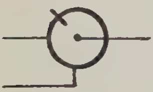
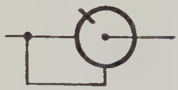
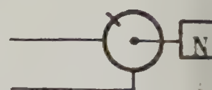
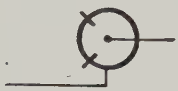
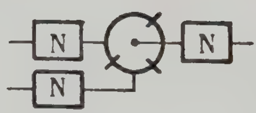
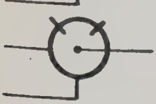
3. Emil L. Post, The Two-Valued Iterative Systems of Mathematical Logic (Princeton: Princeton University Press, 1941). (Annals of Mathematics Studies, No. 5) pp. 105-118.

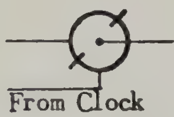
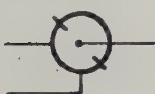
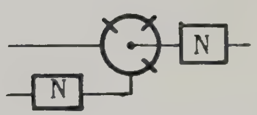
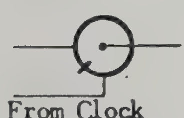
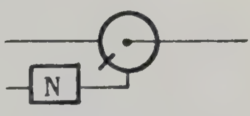
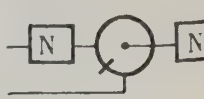
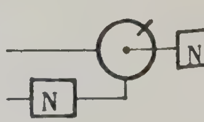
4. We assume that K,N and A,N are complete. Proof of this is contained in Tenny Lode, 'The Realization of a Universal Decision Element', Journal of Computing Systems, Vol.1, No. 1 (June, 1952) pp. 21-22, when combined with the result that NSpq = Apq, NDpq = Kpq and NNp = p. See Goodell, 'The Logic of Decision Elements', Advance Release, (St. Paul, Minnesota Electronics Corp.), 1953, p.7. For explanation of the symbols used, see the last paper mentioned and Lode, op. cit. It should be noted that where the clock pulse is not available or where 0 is represented by a negative pulse, our theorem gives other results than indicated in Table 1. The theorem, of course, remains true. Cf. Goodell, 'Foundations of Computing Machinery, Part II' Journal of Computing Systems, Vol.1, No.2, (1953), p.88.

WILLOW RUN RESEARCH CENTER
YPSILANTI, MICHIGAN

TABLE I

Minimal complete sets of Decision Elements

	Circuit symbols	synthesis of N	synthesis of K	synthesis of A
1.	 ≡ D			
2.	 ≡ S			
3.	 ≡ K  ≡ P		Given	
4.	 ≡ K  ≡ J		Given	As in 3.
5.	 ≡ A  ≡ P	As in 3.		Given
6.	 ≡ A  ≡ J	As in 4.	As in 5	Given

	Circuit Symbols	synthesis of N	synthesis of K	synthesis of A
.	 ≡ E  ≡ K		Given	As in 3.
.	 ≡ K  ≡ R		Given	As in 3.
9.	 ≡ A  ≡ E	As in 7.	As in 5	Given
10.	 ≡ A  ≡ R	As in 8	As in 5	Given
11.	 ≡ C			
12.	 ≡ L			
13.	 ≡ H			
14.	 ≡ T			

SINGLE AXIOMS FOR THE SYSTEMS (C,N), (C,O) AND (A,N) OF THE TWO-VALUED PROPOSITIONAL CALCULUS

CAREW A. MEREDITH

1.

The axioms I develop are:-

I CCCCCpqCNrNsrtCCtpCsp

II CCCCCpqCrOstCCtpCrp

III CCCpqArAstCCspArAtp

where, in (III), 'C' is an abbreviation for 'AN'.

In all three the binary functor occurs nine times, and only one variable occurs three times. (I) and (II) have a common C-structure; (I) and (III) have a common multiplicity of the variables, viz., three 'p's, one 'q' and two each of 'r', 's' and 't'.

Previous axioms for (C,N) and (C,O), only two letters longer than the ones given here, have been given by Łukasiewicz,[1] viz.:

(i) CCCpqCCCNrNstrCuCCrpCsp

(ii) CCCpqCCNrsCNttCCtpCuCrp

(iii) CCCpqCCCrOstCuCCtpCrp

(iv) CCCpqCCOrCstCuCCtpCsp

I know of no previously given axiom, of reasonable length, for the third system.

I shall state, without proof, another (C,O) and two other (A,N) axioms. I have found no other 21-letter (C,N) axiom.

IV CCCpqCCOrsCCspCtCup

V CCCpqArAstCCtsArAps

VI CCCpqArAstCCrpAtAsp

(IV) has a development very similar to that of CCCpqrCCrpCsp,

(Received March 5, 1953)

the Łukasiewicz implicational axiom[2], and, like it, has only two letters occurring more than once. The deduction remains intact if, for 0, we substitute a suitable function of any or all of the variables, <u>excluding</u> 'q'; and we may thus obtain a variety of 23-letter (C,N) axioms.

(V) and (VI) have developments very similar to that of (III), but heavier. They have all, in effect, a period of 3; applying any one to an expression Q, and obtaining in succession Q_1, Q_2, Q_3, Q_4, Q_4 is identical with Q_1. In a similar sense (III) and (V) are each the square of the other.

I have almost completed a proof that no single axiom of (C,O) can contain less than 19 letters; and I think this should, with modifications, be convertible into a similar proof for (C,N) with 21 letters. A single (A,N) axiom must contain at least five 'N's, every thesis of four 'N's or less satisfying one or other of the matrices:

A	1	2	3	N
1	1	1	1	3
2	1	1	1	2
3	1	1	3	1

A	1	2	3	N
1	1	1	1	3
2	1	3	3	2
3	1	3	3	1

in each case '1' being the designated value. But there are at least ten Fregoid two-axiom bases, e.g.:

VII CpApq, CApAqrCCqpArp

consisting of a one-N and a three-N axiom.

I use the following standard bases:

For (C,N), Łukasiewicz:

CCpqCCqrCpr
CpCNpq
CCNppp

For (C,O), Tarski-Bernays:

CCpqCCqrCpr
CpCqp

CCCpqpp
COp

For (A,N), Whitehead and Russell:

CCqrCApqApr
CApqAqp
CpAqp
CAppp

My indebtedness to Professor Łukasiewicz for the general form of these axioms will be very evident.

2.

1 CCCCCpqCNrNsrtCCtpCsp
1 p/Cpq, q/CNCNtNrNs, r/t, s/r, t/CCtpCsp x C1 r/CNtNr - 2

2 CCCCtpCspCpqCrCpq
1 p/t, q/Np, r/CNpq, s/p, t/CrCNpq x C2 p/Np, s/NCNpq - 3

3 CCCrCNpqtCpt
1 p/r, r/p, t/Cpp x C3 r/Crq, q/Ns, t/p - 4

4 CCCpprCsr
4 p/Cpp, r/CsCpp, s/q x C4 r/Cpp - 5

5 CqCsCpp
1 p/r, s/q, t/CsCpp x C5 q/CCCrqCNrNqr - 6

6 CCCsCpprCqr
1 s/r, t/Cqr x C6 s/Cpq, p/Nr - 7

7 CCCqrpCrp
7 q/CCCpqCNrNsr, r/t, p/CCtpCsp x C1 - 8

8 CtCCtpCsp
8 t/CCCrCNpqtCpt, p/u x C3 - 9

9 CCCCCrCNpqtCptuCsu
1 p/CrCNNpq, q/Nt, r/p, s/t, t/Csp x C9 p/Np, t/Nt, u/p - 10

10 CCCspCrCNNpqCtCrCNNpq

10 s/CCCqqCNrNNNpr, r/Cpq, t/1 x C1 p/q, s/NNp, t/p - C1 - 11

11 CCpqCNNpq

7 p/CCCqrpCsp x C8 t/Cqr - 12

12 CrCCCqrpCsp

1 t/CCCtCCCpqCNrNsruCvu x C12 r/CCCpqCNrNsr, q/t, p/u, s/v - 13

13 CCCCCtCCCpqCNrNsruCvupCsp

1 p/CtCCCpqCNrNsr, q/Nu, r/p, s/u, t/Csp x C13 u/Nu, v/Np - 14

14 CCCspCtCCCpqCNrNsrCuCtCCCpqCNrNsr

14 t/CCspr, u/1 x C8 t/Csp, p/r, s/CCpqCNrNs - C1 - 15

15 CCCsprCCCpqCNrNsr

15 r/CCCtCsprCur x C12 r/Csp, q/t, p/r, s/u - 16

16 CCCpqCNCCCtCsprCurNsCCCtCsprCur

1 r/CNCCCtCsCpqrCurNs, s/v, t/CCCtCsCpqrCur x C16 p/Cpq, q/CNCNCCCtCsCpqrCurNsNv - 17

17 CCCCCtCsCpqrCurpCvp

1p/CtCsCpq, q/Nr, r/p, s/r, t/Cvp x C17 r/Nr, u/Np - 18

18 CCCvpCtCsCpqCrCtCsCpq

18 v/CCCCpqqCNrNsr, t/CpCpq, r/1 x C1p/Cpq, t/p - C1 - 19

19 CCpCpqCsCpq

19 p/CpCpq, q/Cpq, s/1 x C19 s/CpCpq - C1 - 20

20 CCpCpqCpq

3 r/Np, t/CNpq x C20 p/Np - 21

*21 CpCNpq

8 t/CCpqCNNpq, p/r x C11 - 22

22 CCCCpqCNNpqrCsr

20 p/CCCpqCNNpqr, q/r x C22 s/CCCpqCNNpqr - 23

23 CCCCpqCNNpqrr

1 q/Ns, r/Np, t/Np x C23 q/Ns, r/Np - 24

24 CCNppCsp

20 p/CNpp, q/p x C24 s/CNpp - 25

*25 CCNppp

4 p/r, r/CNNrr x C11 p/r, q/r - 26

26 CsCNNrr

8 t/CsCNNrr, s/q x C26 - 27

27 CCCsCNNrrpCqp

20 p/CCsCNNrrp, q/p x C27 q/CCsCNNrrp - 28

28 CCCsCNNrrpp

11 p/CCsCNNrrp, q/p x C28 - 29

29 CNNCCsCNNrrpp

12 r/CNNCCsCNNrrpp, p/t, s/u x C29 - 30

30 CCCqCNNCCsCNNrrpptCut

1 p/q, r/NCCsCNNrrNp, s/p, t/CuNCCsCNNrrNp x C30 q/Cqq,
p/Np, t/NCCsCNNrrNp - 31

31 CCCuNCCsCNNrrNpqCpq

31 u/CsCNNrr, q/NCCsCNNrrNp x C28 p/NCCsCNNrrNp - 32

32 CpNCCsCNNrrNp

12 r/CpNCCsCNNrrNp, p/t, s/u x C32 - 33

33 CCCqCpNCCsCNNrrNptCut

20 p/CCqCpNCCsCNNrrNpt, q/t x C33 u/CCqCpNCCsCNNrrNpt - 34

34 CCCqCpNCCsCNNrrNptt

1 p/q, r/p, s/CCsCNNrrNNp, t/p x C34 q/Cqq, p/Np, t/p - 35

35 CCpqCCCsCNNrrNNpq

35 p/Cpq, q/CCCsCNNrrNNpq, s/t, r/u x C35 - 36

36 CCCtCNNuuNNCpqCCCsCNNrrNNpq

1 p/t, r/NNCpq, s/Cpq, t/CCCsCNNrrNNpq x C36 t/Ctq, u/NCpq - 37

37 CCCCCsCNNrrNNpqtCCpqt

37 s/Crq, r/Np, t/CCqrCpr x Cl p/r, r/NNp, s/p, t/q - 38

*38 CCpqCCqrCpr

3.

1 CCCCCpqCrOstCCtpCrp

1 p/Cpq, q/CrO, r/s, s/t, t/CCtpCrp x Cl s/CsO - 2

2 CCCCtpCrpCpqCsCpq

1 p/t, q/O, s/COq, t/CsCOq x C2 p/O - 3

3 CCCsCOqtCrt

3 s/CCCpqCrOs, t/CCCOqpCrp, r/l x Cl t/COq - Cl - 4

4 CCCOqpCrp

2 t/O, r/q, q/Cqp, s/l x C4 q/p, p/Cqp, r/p - Cl - 5

*5 CpCqp

1 t/CtCCCpqCrOs x C5 p/CCCpqCrOs, q/t - 6

6 CCCtCCCpqCrOspCrp

1 p/s, r/CCpqCrO, s/p, t/Crp x C6 t/Csq, s/O - 7

7 CCCrpsCCCpqCrOs

7 r/CCCpqCrOs, p/t, s/CCtpCrp, q/u x Cl - 8

8 CCCtuCCCCpqCrOsOCCtpCrp

7 r/Ctu, p/CCCCpqCrOsO, s/CCtpCrp, q/v x C8 - 9

9 CCCCCCCpqCrOsOvCCtuOCCtpCrp

7 r/CCCCCpqCrOsOv, p/CCtuO, s/CCtpCrp, q/w x C9 - 10

10 CCCCtuOwCCCCCCpqCrOsOvOCCtpCrp

1 p/Ctu, q/O, r/w, s/CCCCCCpqCrOsOvO, t/CCtpCrp x C10 w/CwO - 11

11 CCCCtpCrpCtuCwCtu

11 t/Ctp, p/Crp, r/t, u/CtCrp, w/1 x C11 u/Crp, w/Ctp - C1 - 12

12 CCtpCtCrp

12 t/CCtuCCCCpqCrOsO, p/CCtpCrp, r/v x C8 - 13

13 CCCtuCCCCpqCrOsOCvCCtpCrp

1 p/s, r/u, s/CCCCpqCrOsO, t/CvCCCsqpCrp x C13 t/Csq, u/CuO - 14

14 CCCvCCCsqpCrpsCus

7 r/CvCCCsqpCrp, p/s, s/Cus, q/w x C14 - 15

15 CCCswCCvCCCsqpCrpOCus

7 r/Csw, p/CCvCCCsqpCrpO, s/Cus, q/y x C15 - 16

16 CCCCCvCCCsqpCrpOyCCswOCus

1 p/CvCCCsqpCrp, q/O, r/w, s/CCswO, t/Cus x C16 y/CwO - 17

17 CCCusCvCCCsqpCrpCwCvCCCsqpCrp

17 u/CCCCrpqCCCsqpOs, v/CsCrp, w/1 x C1 p/Crp, r/CCsqp, t/s - C1 - 18

18 CCsCrpCCCsqpCrp

18 s/p x C5 q/r - 19

19 CCCpqpCrp

1 s/Cpq, t/CsCpq x C19 p/Cpq, q/CrO, r/s - 20

20 CCCsCpqpCrp

1 p/s, r/p, s/p, t/Crp x C20 s/Csq, q/O - 21

21 CCCrpsCps

21 r/CCCpqCrOs, p/t, s/CCtpCrp x C1 - 22

22 CtCCtpCrp

18 s/t, r/Ctp, p/Crp x C22 - 23

23 CCCtqCrpCCtpCrp

23 p/Ctq x C5 p/Ctq, q/r - 24

24 CCtCtqCrCtq
11 r/Ctp, u/CCtpp, w/1 x C24 t/Ctp, q/p, r/t - C1 - 25

25 CtCCtpp
21 s/CCCrpqq x C25 t/Crp, p/q - 26

26 CpCCCrpqq
23 t/p, q/s, r/CCrCpsq, p/q x C26 p/Cps - 27

27 CCpqCCCrCpsqq
27 p/Cpq, q/CCCrCpsqq, r/t, s/u x C27 - 28

28 CCCtCCpquCCCrCpsqqCCCrCpsqq
1 p/t, r/Cpq, s/CCCrCpsqq, t/CCCrCpsqq x C28 t/Ctq, u/0 - 29

29 CCCCCrCpsqqtCCpqt
29 r/Crq, s/0, t/CCqrCpr x C1 p/r, r/p, s/q, t/q - 30

*30 CCpqCCqrCpr
24 t/CCpqp, q/p, r/1 x C19 r/CCpqp - C1 - 31

*31 CCCpqpp
31 p/C0p x C4 q/p, p/q, r/0 - 32

*32 C0p

4.

1 CCCpqArAstCCspArAtp
1 p/Cpq, q/ArAst, r/NCsp, s/r, t/Atp x C1-2

2 CCrCpqCCspAAtpCpq
1 p/r, q/Cpq, r/NCsp, s/Atp, t/Cpq x C2-3

3 CCAtprCCspACpqr
1 p/Atp, q/r, r/NCsp, s/Cpq, t/r x C3-4

4 CCCpqAtpCCspArAtp
4 q/Aqp, t/q, r/Np x 5

5 CCCpAqpAqpCCspCpAqp

4 p/CpAqp, q/Aqp, t/NCsp, s/t x C5-6

6 CCtCpAqpArCCspCpAqp

4 p/CCspCpAqp, q/CpAqp, t/r, s/CCpAqpAqp, r/t x C6
t/CCspCpAqp - C5-7

7 AtArCCspCpAqp

7 t/N1, r/N1 x C1-C1-8

8 CCspCpAqp

4 p/Aqp, q/p, t/Np x C8 s/Aqp -9

9 CCsAqpArCpAqp

4 p/CpAqp, q/Aqp, t/r, s/Csp, r/t x C9 s/CpAqp -C8-10

10 AtArCpAqp

10 t/N1, r/N1 x C1-C1-11

*11 CpAqp

4 p/ACpqp, q/p, t/NCsp, s/t x C3 t/Cpq, r/p -12

12 CCtACpqpArCCspACpqp

12 t/p, r/N1 x C11 q/Cpq -C1-13

13 CCspACpqp

4 q/p, t/Cpq x C13 s/p -14

14 CCspArACpqp

1 p/s, q/p, s/Cpq, t/p x C14-15

15 CCCpqsArAps

1 p/Cpq, q/s, s/p, t/s x C15-16

16 CCpCpqArAsCpq

16 q/p, r/N1 x C11 q/Np -C1-17

17 AsCpp

17 s/N1 x C1-18

18 Cpp

1 r/NApq, s/p, t/q x C17 s/NCpq, p/Apq -C18-19

*19 CApqAqp

11 p/CApqAqp, q/r x C19-20

20 ArCApqAqp

1 p/r, r/NApq, s/q, t/p x C20 r/NCrq -21

*21 CCqrCApqApr

1 q/r, r/NApp, s/p, t/r x C21 q/p -C18-22

22 CAppArp

16 p/App, q/p, r/N1, s/N1 x C22 r/NApp -C1-C1-23

*23 CAppp

NOTES

1) (i) and (iv) were published, without proofs, in 1936 and 1933 respectively. The other two were never published.

2) For proof, see J. Łukasiewicz, The Shortest Axiom of the Implicational Calculus of Propositions, Proceedings of the Royal Irish Academy, Vol. 52, A 3 (1948).

TRINITY COLLEGE
DUBLIN, IRELAND.

A FORMALIZATION OF SOBOCIŃSKI'S THREE-VALUED IMPLICATIONAL PROPOSITIONAL CALCULUS

ALAN ROSE

Sobociński has recently formalized[1] a three-valued propositional calculus with two designated truth-values. The primitive functors, C and N, have the truth-tables given below, 1 and 2 being the designated truth-values.

	Cpq	q = 1	2	3	Np
	1	1	3	3	3
p	2	1	2	3	2
	3	1	1	1	1

He uses the axioms:

1 CCpqCCqrCpr
2 CpCCpqq
3 CCpCpqCpq
4 CpCqCNqp
5 CCNpNqCqp

together with the rules of substitution and modus ponens.

He raises[2] the question whether the corresponding system with C as its only primitive can be formalized, and if so, what is its degree of completeness. It has been shown[3] that if such a weakly complete formalization exists and it uses the above rules of procedure and no others, then its degree of completeness is 3. In the present paper we shall consider a formalization of the system.

(Received March 26, 1953)

A FORMALIZATION OF SOBOCINSKI'S PROPOSITIONAL CALCULUS

We shall use the axioms

A1 CCpqCCqrCpr

A2 CpCCpqq

A3 CCpCpqCpq

A4 CCCpCqCCqCrrpCrrCrr

A5 CCCCCpCrrCqCrrCqpCrrCrr

A6 CCCpCrrCrrCCCCpqCrrCrrCCqCrrCrr

together with three rules of procedure. These are the rules of substitution and modus ponens, and the rule 'If no propositional variable occurs in both the formulae P, Q and CCPCQQCQQ is a theorem, then P is a theorem.'

Theorem 1 The formalization is weakly complete.

We define the class of a formula of the original system as follows:

(i) If the propositional variable p does not occur in the formula P then CCpCPPCPP is of class p.

(ii) If CCRCQQCQQ is of class P then CCCRCQQCQQCQQ is of class NP.

(iii) If CCRCTTCTT, CCSCTTCTT are of classes P, Q respectively then CCCRSCTTCTT is of class CPQ.

Lemma: If P is provable in the original system and P* is of class P, then P* is provable in the present system.

Let us suppose that P is provable in the original system by n applications of the rules of procedure. We shall prove the lemma by strong induction on n.

If n = 0 then P is one of the original axioms. If it is one of the first three, then P* is of the form CCPCQQCQQ. But since the first three axioms are unchanged in the present system, P is provable. Applying the substitution rule to A2, we deduce CPCCPCQQCQQ. Applying modus ponens to the last two formulae we deduce CCPCQQCQQ, i.e., P*. If P is one of the last two original axioms, then either P* is A4 or A5 or P* can be deduced from one of these axioms by substituting for r by the substitution rule. Thus, the lemma holds when n = 0.

We now assume the lemma for 0, 1, ..., n and prove it for n + 1. If the (n + 1)TH stage in the deduction of P is the substitution of a formula R for a propositional variable p occurring

in a formula Q, then Q is provable after n applications of the rules. It follows from our induction hypothesis that if Q* is of class Q, then Q* is provable in the present system. Thus, we can prove a formula of the form CCSCqqCqq of class Q, where p,q are distinct propositional variables and q does not occur in P. Substituting R for p in CCSCqqCqq we deduce a formula CCTCqqCqq of class P. Thus, if P* is CCUCVVCVV we can deduce P* by substituting V for q in CCTCqqCqq.

If the $(n+1)^{TH}$ stage in the deduction of P in the original system is the application of modus ponens to formulae Q, CQP then Q, CQP are provable in the original system after not more than n applications of the rules. Hence, by our induction hypothesis, all formulae of classes Q, CQP are provable in the present system. Let p be a propositional variable which does not occur in P or in Q, and let CCRCppCpp, CCSCppCpp be of classes P, Q respectively. Then CCCSRCppCpp is of class CQP. Applying the substitution rule to A6 we deduce CCCSCppCppCCCCSRCppCppCCRCppCpp. Since the formulae CCSCppCpp, CCCSRCppCpp are provable we obtain CCRCppCpp after two applications of modus ponens. Then either this formula is P* or P* can be deduced by substituting for p in this formula by the substitution rule. Thus, the lemma is proved.

Proof of the main theorem. If the only functor occurring in P is C, then P* is of the form CCPCQQCQQ where no propositional variable occurs in both P and Q. But if the truth-value of P is always designated, P is provable in the original system. It follows from the lemma that P* is provable in the present system. Applying the third rule of procedure we deduce P.

Theorem 2 The formalization is strongly complete.

Let us suppose that we adjoin to the axioms the unprovable formula P. If P is not an identically true formula of the 2-valued propositional calculus, let us denote P by $\phi(p_1, p_2, \ldots, p_n)$ and let us suppose that P takes the truth-value 3 when $p_{i_1}, p_{i_2}, \ldots, p_{i_j}$ take the truth-value 1 and $p_{i_{j+1}}, p_{i_{j+2}}, \ldots, p_{i_n}$ take the truth-value 3. Let us substitute Cpp for $p_{i_1}, p_{i_2}, \ldots, p_{i_j}$ and let us substitute p for $p_{i_{j+1}}, p_{i_{j+2}}, \ldots, p_{i_n}$. Let us denote the resulting formula by $\psi(p)$. Then $\psi(p)$ becomes provable in the formalization and takes the same truth-value as p. It follows from the weak completeness that $C\psi(p)p$ is provable. Applying modus ponens to this $\psi(p)$ we deduce p. Any formula can then be deduced by applying the substitution rule to p.

If P is an identically true formula of the 2-valued propositional calculus, but not of the present system, it follows from a theorem of the author[4] that every identically true formula of the 2-valued propositional calculus becomes provable. Thus, all formulae of the form CCQCRRCRR become provable. In order to prove any formula Q let us choose R so that no propositional variable occurs in both Q and R. Applying the third rule of procedure we deduce Q. Thus, the theorem is proved.

NOTES

1) Bolesław Sobociński: Axiomatization of a Partial System of Three-Value Calculus of Propositions. The Journal of Computing Systems, Vol. 1, 1952.

2) Op. cit.

3) Alan Rose: Le degré de saturation du calcul propositionnel implicatif à trois valeurs de Sobociński. Comptes-rendus de l'Académie des Sciences, Paris, Vol. 235, 1952.

4) Op. cit.

THE UNIVERSITY OF NOTTINGHAM
NOTTINGHAM, ENGLAND.

A SINGLE AXIOM OF POSITIVE LOGIC

CAREW A. MEREDITH

I deduce the Łukasiewicz base

CpCqp

CCpCqrCCpqCpr

from the axiom

CCCpqrCsCCqCrtCqt

I do not know if this is the shortest axiom of the system. The occurrence of two letters of simplification, 's' and, in an extended sense, 'p', seems to indicate superfluity; and the deduction is very easy. On the other hand, the Łukasiewicz axiom of the two-valued implicational calculus contains two singly occurring variables.

1. CCCpqrCsCCqCrtCqt
   1 p/Cpq, q/r, r/CsCCqCrtCqt, s/p, t/u x C1-2

2. CpCCrCCsCCqCrtCqtuCru
   2 p/1 x C1-3

3. CCrCCsCCqCrtCqtuCru
   3 r/CqCrt, s/r, q/s, t/Cqt, u/CrCsCqt x C2 p/CqCrt, u/CsCqt -4

4. CCqCrtCrCsCqt
   1 p/q, q/Crt, r/CrCsCqt, s/p, t/u x C4-5

5. CpCCCrtCCrCsCqtuCCrtu
   3 r/s, s/Crt, q/r, t/Cqt, u/CCrtCrCqt x C5 p/s, u/CrCqt -6

6. CsCCrtCrCqt
   6 s/1 x C1-7

7. CCrtCrCqt
   1 p/r, q/t, r/CrCqt, t/p x C7-8

(Received March 5, 1953)

8. CsCCtCCrCqtpCtp
   3 r/q, s/t, q/r, u/CtCrt x C8 s/q, p/Crt -9

9. CqCtCrt
   9 q/1, t/p, r/q x C1-10

*10. CpCqp
   3 r/CCpqr, u/CCqCrtCqt x C1 s/CsCCqCCCpqrtCqt -11

11. CCCpqrCCqCrtCqt
   11 r/CtCst, t/p x C9 q/Cpq, r/s -12

12. CCqCCtCstpCqp
   12 q/CCtCstCCtCstp x C12 q/CtCst -13

13. CCCtCstCCtCstpp
   11 p/CtCst, q/CCtCstp, r/p, t/q x C13-14

14. CCCCtCstpCpqCCCtCstpq
   10 p/CCCpqrCCqCrtCqt, q/s x C11-15

15. CsCCCpqrCCqCrtCqt
   11 p/CCtCstp, q/Cpq, r/CCCtCstpq, t/CCpCqrCpr x C14-C15 s/Cpq, p/CtCst, q/p, r/q, t/r -16

16. CCpqCCpCqrCpr
   4 q/Cpq, r/CpCqr, t/Cpr x C16-17

17. CCpCqrCsCCpqCpr
   12 q/CpCqr, p/CCpqCpr x C17 s/CtCst -18

*18. CCpCqrCCpqCpr

It is, perhaps, worth noting that (16), instead of (18), with (5) gives a sufficient two-axiom base.

TRINITY COLLEGE
DUBLIN, IRELAND.

NOTE ON A MODAL SYSTEM OF FEYS-VON WRIGHT

BOLESŁAW SOBOCIŃSKI

In this note I intend to show that the System T proposed by R. Feys[1] in 1937 and the System M recently published by G. H. von Wright[2] are inferentially equivalent, and, moreover, that in this system (T or M) there are infinitely many modalities. Some other considerations are given toward the end.

Following R. Feys[3], I shall use here a modification of Łukasiewicz's symbolism of propositional calculus in which 'C', 'K', 'A', 'E', 'N' possess the ordinary meaning, and 'L' means 'necessity', and 'M' 'possibility'. A familiarity with this symbolism, with the bi-value propositional calculus and with the Lewis's modal calculi is presupposed.

As abbreviations I shall use the following symbol: '$\{A;B;\ldots\} \rightleftarrows \{P;Q;R;\ldots\}$' which will denote that a set of postulates A;B; ... is inferentially equivalent to a set of P;Q;R; ... Similarly: '$\{A;B;\ \ldots\ \} \rightarrow \{P;Q;R;\ \ldots\}$' will denote that from a set of postulates A;B; ... we can deduce the theses P;Q;R; ... Where it cannot involve a misunderstanding I shall use also the similar symbolism for the names of the systems, e.g.: $\{S3\} \rightarrow \{S2\}$.

1. We shall consider some sets of postulates taken from the class of the following assumptions:

I. A complete axiomatic of the bi-value propositional calculus with the suitable rules of procedure.

But these rules are adjusted either:

I.a. To the functor L and the definition:

Def. I $\quad Mp = NLNp$

or

I.b. To the functor M and the definition:

Def. II $\quad Lp = NMNp$

II. Two special rules of procedure:

Rule I. If an expression α is a thesis of a considerated

(Received January 31, 1953)

system, then also $L\alpha$ is a thesis of this system.

Rule II. If an expression $E\alpha\beta$ is a thesis of a considerated system, then also $EM\alpha M\beta$ is a thesis of this system.

III. The following modal theses:

G1	CLpp	W1	CpMp
G2	CLCpqCLpLq	W2	EMApqAMpMq
G3	CLpLLp	W3	CMMpMp
G4	CNLpLNLp	W4	CMNMpNMp

1.1 In 1933 K. Gödel has observed[4] that:

1.1.1 A set of postulates {I.a; R I; G1; G2; G3} $\rightleftarrows$ {S4}.
1.1.2 A set of postulates {I.a; R I; G1; G2; G4} $\rightleftarrows$ {S5}.

1.2 In the now published book G. H. von Wright gives the axioms of his modal systems M, M′, M″[5]:

1.2.1 For the system M: {I.b; R I; R II; W1; W2}. He also has proved that: {M} $\rightarrow$ {S2} and that the System M does not contain S3.

1.2.2 For the System M′: {I.b; R I; R II; W1; W2; W3}. He also has proved that: {M′} $\rightleftarrows$ {S4}.

1.2.3 For the System M″: {I.b; R I; R II; W1; W2; W4}. He also has proved that: {M″} $\rightleftarrows$ {S5}.

1.3 In 1937 R. Feys proposed[6] a system of modal logic called T: {I.a; R I; G1; G2}, supposing without a proof that this system possesses the infinitely many modalities.

As far as I know, little attention was paid to this proposal, and a place for the System T is not yet established in the field of Lewis's calculi. Only W. T. Parry has shown[7] that: If a thesis

G5 CLCpqLCLpLq

is provable in this system, then {T} $\rightleftarrows$ {S4}. If G5 is not provable, then this system neither includes nor is included in S3. Also, due to Parry, we know that the addition of the rule RI (which is provable in S4) to S3 converts it into S4. Thus, from these results of Parry follows: {I.a; RI; G1; G2; G5} $\rightleftarrows$ {S4} and {S3; RI} $\rightleftarrows$ {S4}.

The question of how many modalities the System T possesses remained open.

1.4 In the 1950 paper[8] of Moh Shaw-Kwei the role of the rule RI in the field of Lewis's systems is discussed. He states that the addition of RI to S1 and to S2 yields the same results. A system obtained in this way contains S2, is independent of S3 and is contained in S4.

2. In order to prove that there are infinitely many modalities in the System T we only have to show that a matrix which J.C.C. McKinsey has constructed to prove the same for S2[9] also satisfies the System T. It is evident that the postulates I.a; G1 and G2 are provable in S2.[10] Then, McKinsey's matrix satisfies them. If an expression α is a thesis provable from {I.a; G1; G2}, then, according to this matrix $\alpha = V$ (V is a designated value). But $L\alpha = LV = NMNV = NM\Lambda = N\Lambda = V$. Thus, the rule RI is satisfied also by this matrix, and, therefore, the proof of McKinsey holds not only for S2 but also for T.

Hence, this System T is an intermediary system between S2 and S4, and different from S3. It cannot be equivalent with S3 or S4 because these two possess a finite number of modalities[11], and it is stronger than S2, because in S2 the rule RI evidently is not provable[12]. Then, it is evident that: $\{T\} \rightleftarrows \{S2;\ RI\} \rightleftarrows \{S1;\ RI\}$[13].

In order to establish a place for the System T it would be worthwhile to resolve a question: whether in the System T it is possible to substitute in an adequate way the rule RI by a finite number of the axioms. This problem remains open.

3. The systems T and M are inferentially equivalent.

3.1 From the axioms of T: {I.a; RI; G1; G2} we can prove:[14]

A1	EMpNLNp	[I.a; Def I]
A2	CLNNpLp	[I.a; RI; G2]
A3	CLpLNNp	[I.a; RI; G2]
A4	ELpLNNp	[I.a; A2; A3]
A5	ELpNMNp	[I.a; A1; A4]

Having A5 we have not only our assumption I.a, but also I.b.

W1	CpMp	[I.a; G1; A1]
A6	CLNApqLNp	[I.a; RI; G2]
A7	CMpMApq	[I.a; A6; A1]
A8	CMqMApq	[I.a; RI; G2; A1]
A9	CLNqCLNpLNApq	[I.a; RI; G2; I.a; G2]
A10	CMApqAMpMq	[I.a; A9; A1]
W2	EMApqAMpMq	[I.a; A10; A7; A8]

A proof of the rule RII: If an expression $E\alpha\beta$ is a thesis of the System T, then also $EM\alpha M\beta$ is a thesis of this system.

α) $\{E\alpha\beta\} \longrightarrow \{CM\alpha M\beta\}$

Proof:

If:

a) $E\alpha\beta$ [Assumption]

then:

b) $CN\beta N\alpha$ [I.a; a]
c) $CLN\beta LN\alpha$ [b; RI; G2]
$CM\alpha M\beta$ [I.a; c; A1]

β) $\{E\alpha\beta\} \longrightarrow \{CM\beta M\alpha\}$ [Similarly; I.a; RI; G2; A1]
γ) $\{E\alpha\beta\} \longrightarrow \{EM\alpha M\beta\}$ [I.a; α; β]

Thus: {I.a; RI; G1; G2} → {I.b; RII; W1; W2} and, therefore: {T} → {M}.

3.2 From the axioms of M: {I.b; RI; RII; W1; W2} we can prove:

B1	ELpNMNp	[I.b; Def. II]
B2	EMpMNNp	[I.b; RII]
B3	EMpNLNp	[I.b; B1; B2]

Having B3 we have not only our assumption I.b but also I.a.

G1	CLpp	[I.b; W1; B1]
B4	EMNCpqMAKKpNqrKKpNqNr	[I.b; RII]
B5	EMNCpqAMKKpNqrMKKpNqNr	[I.b; B4; W2]
B6	ELCpqKNMKKpNqrNMKKpNqNr	[I.b; B5; B1]
B7	ELCqrKNMKKpqNrNMKKNpqNr	[Similarly; I.b; RII; W2; B1]
B8	ELCprKNMKKpqNrNMKKpNqNr	[Similarly; I.b; RII; W2; B1]
B9	CLCpqCLCqrLCpr[15]	[I.b; B6; B7; B8]
B10	EMNpMNCNpp	[I.b; RII]
B11	ELpLCNpp	[I.b; B10; B1]
B12	EMNCpqMNCNqNp	[I.b; RII]
B13	ELCpqLCNqNp	[I.b; B12; B1]
B14	CLCpqCLpLCNpq	[I.b; B11; B9]
B15	CLCpqCLpLCNqq	[I.b; B14; B13; B9]
G2	CLCpqCLpLq[15]	[I.b; B15; B11]

Thus: {I.b; RI; RII; W1; W2} → {I.a; G1; G2} and, therefore: {M} → {T}.

3.3 From 3.1 and 3.2 follows that the systems T and M are inferentially equivalent.

4. In the axiomatic of the System M″ the rule RI is redundant.

4.1 From the set of postulates {I.b; RII; W1; W4} we can prove:

E1	ELpNMNp	[I.b; Def. II]
E2	CMpNMNMp	[I.b; W4]
E3	CMpLMp	[I.b; E2; E1]
E4	MCpp	[W1; I.b]
E5	LMCpp	[E3 ; E4]

Now we prove the following rule:

Rule III. If an expression $E\alpha\beta$ is a thesis of this considerated system, then also $EL\alpha L\beta$ is a thesis of this system.

Proof:

If:

a) $E\alpha\beta$ [Assumption]

then:

b)	$EN\alpha N\beta$	[I.b; a]
c)	$EMN\alpha MN\beta$	[b; RII]
d)	$ENMN\alpha NMN\beta$	[I.b; c]
	$EL\alpha L\beta$	[I.b ; b ; E1]

Since we have the rule RII:

If an expression α is a thesis of this system, then also $L\alpha$ is a thesis of this system.

Proof:

If:

a) α [Assumption]

then:

b)	$EMCpp\alpha$	[I.b; a; E4]
c)	$ELMCppL\alpha$	[b; RIII]
	$L\alpha$	[c; E5]

4.2 Hence, we have proved that {I.b; RI; RII; W1; W2; W4} $\rightleftarrows$ {I.b; RII; W1; W2; W4}. Using the entirely similar methods we can prove also that in the axiomatics of the systems M and M′ we can replace the rule RI by, e.g., the thesis: LCpp.[16]

5. The difference between the System T and the Systems S2 and S3 appears much more obvious if we try to add to the System T either the axiom

C13 MMp or the axiom C14 LMMp

Both these theses are consistent with S2 or S3, and their addition to these systems generates the new systems known as S6, S7 and S8[17]; whereas, the addition of one of these theses, C13 or C14, to the System T leads to the contradictory. Namely:

5.1 In the field of the System T we can prove the following additional theses:

A11	LNKpNp	[I.a; RI]
A12	ENKpNpLNKpNp	[I.a; A11]
A13	EKpNpMKpNp	[I.a; A12; A1]
A14	EMKpNpMMKpNp	[A13; RII]
A15	EMMKpNpKpNp	[I.a; A13; A14]
A16	LEKpNpMKpNp	[A13; RI]
A17	LCMKpNpKpNp	[I.a; A13; RI]

5.2 Now, evidently if we add the thesis C14 to the System T, then we have also C13 (by G1). But C13 and A15 give a contradiction.

Thus, thesis A16, unprovable in S3 but provable in S4,[18] is also a thesis of the System T. S. Halldén has shown[19] that the addition of A17 to S3 converts this system into S4 and that the addition of the negation of A17, i.e., of the thesis

F1 NLCMKpNpKpNp

to S3 gives S7. In respect to the System T, A17 is its thesis and, therefore, F1 cannot be added. These examples show in the expressive way the differences between the Systems T and S2 or S3.

6. In this short note there is given a superficial analysis of von Wright's systems showing that from the formal point of view the System M (i.e. T) is the most interesting. The problems which can arise in the field of the Systems M′ and M″ are the problems of S4 and S5. When both these systems are well elaborated, we yet know very little about the System M.

NOTE ON A MODAL SYSTEM OF FEYS-VON WRIGHT

NOTES

1) Cf. Feys$_1$, Nr. 25, and Feys$_2$, p.500, note 13. R. Feys called his system a 'system t', but I prefer to use a letter 'T'.

2) Cf. von Wright$_1$, especially Appendix II, pp. 84-90.

3) Cf. Feys$_2$. I do not introduce here a symbol for a strict implication because this functor will not be used.

The definitions of the modal calculi S1-S5 can be found: Lewis-Langford$_1$, pp. 500-501.

4) Cf. Gödel$_1$ and Feys$_2$, No. 16.1-16.24.

5) Cf. von Wright$_1$, Appendix II, pp. 85-90.

6) Cf. Feys$_1$, No. 25 and No. 28.1. Also Feys$_2$, p.500, note 13.

7) Cf. Parry$_1$, p. 148, and Parry$_2$.

8) Cf. Moh Shaw-Kwei$_1$, p. 73.

9) Cf. McKinsey$_1$.

10) Cf. Lewis-Langford$_1$, p. 500, pp. 137-139, p. 163, the thesis 18.42, p. 164, the thesis 18.53. Also Feys$_2$: 6.2; 8.71; 6.42.

11) Cf. Parry$_1$, par. IV-V, and Feys$_2$, 9.75.

12) The rule RI is provable in S4, Cf. Gödel$_1$ and McKinsey-Tarski$_1$, p. 5, Theorem 2.1. It is unprovable in S2 because it is unprovable in S3, Cf. Parry$_1$, p. 148, and S3 contains S2.

13) Cf. Moh Shaw-Kwei$_1$, p. 73.

14) In the proofs given below I do not indicate the bi-value propositional theses which are involved.

15) In the proofs of the theses B9 and G2 I follow von Wright$_1$, p. 88, and Lewis-Langford$_1$, 18.53, respectively.

16) The rule RII is stronger than the rule:

If an expression $LC\alpha\beta$ is a thesis of S2, then also $LCM\alpha M\beta$ is a thesis of S2.

This rule, as C. W. Churchman has proved, holds in S2. Cf. Churchman$_1$, pp. 79-80.

17) The definitions of these systems are given in Halldén$_1$, p. 213.

18) Cf. McKinsey$_2$, p. 126, Theorem 7.

19) Cf. Halldén$_2$, pp. 128-129.

BIBLIOGRAPHY

Churchman$_1$ - C. W. Churchman: On finite and infinite modal systems. The Journal of Symbolic Logic, III (1938), pp. 77-82.

Feys$_1$ - R. Feys: Les logiques nouvelles des modalités. Revue Néo-scholastique de Philosophie, XL (1937), pp. 517-553, and XLI (1938), pp. 217-252.

Feys$_2$ - R. Feys: Les systèmes formalisés des modalités aristotéliciennes. Revue Philosophique de Louvain, XLVIII (1950), pp. 478-509.

Gödel$_1$ - K. Gödel: Eine Interpretation des intuitionistischen Aussagenkalküls. Ergebnisse eines matematischen Kolloquiums, IV (1933), pp. 39-40.

Halldén$_1$ - S. Halldén: Results concerning the decision problem of Lewis's calculi S3 and S6. The Journal of Symbolic Logic, XIV (1949), pp. 213-235.

Halldén$_2$ - S. Halldén: On the semantic non-completeness of certain Lewis calculi. The Journal of Symbolic Logic, XVI (1951), pp. 127-129.

Lewis-Langford$_1$ - C. I. Lewis and C. H. Langford: Symbolic Logic. Second edition, New York.

McKinsey$_1$ - J. C. C. McKinsey: Proof that there are infinitely many modalities in Lewis's system S2. The Journal of Symbolic Logic, V (1940), pp. 110-112.

McKinsey$_2$ - J. C. C. McKinsey: A solution of the decision problem for the Lewis Systems S2 and S4, with an application to topology. The Journal of Symbolic Logic, VI (1941), pp. 117-134.

McKinsey-Tarski$_1$ - J. C. C. McKinsey and A. Tarski: Some theorems about the sentential calculi of Lewis and Heyting. The Journal of Symbolic Logic, XIII (1948), pp. 1-15.

Moh Shaw-Kwei$_1$ - Moh Shaw-Kwei: The deduction theorems and two new logical systems. Methodos, II (1950), pp. 56-75.

Parry$_1$ - W. T. Parry: Modalities in the 'Survey' System of strict implication. The Journal of Symbolic Logic, IV (1939), pp. 137-154.

Parry$_2$ - W. T. Parry: The postulates for 'strict implication'. Mind, XLIII (1934), pp. 78-80.

von Wright$_1$ - G. H. von Wright: An essay in modal logic. Amsterdam, 1951.

THE INSTITUTE OF APPLIED LOGIC
ST. PAUL, MINNESOTA

ANALYTIC MINIMIZATION METHODS I: CONJUNCTIVE FORMS

W.C. CARTER and A.S. RETTIG

The designers of electronic digital computing machines have, following C. E. Shannon[4], focused attention on sentential calculus. The main problems of sentential calculus were formerly regarded as solved. However, to quote Prof. W. V. Quine[3], 'In spite of known procedures, there remains a problem which has proved curiously stubborn, viz., the problem of devising a general mechanical method for reducing any sentential formula to its simplest equivalent.'

The purpose of this paper is to explain a new method of reducing sentential combinations to a simplified conjunctive form. This reduction is accomplished by the application of a series of simple rules. These rules do not require that the function be entered on a chart, nor that the distinguished conjunctive or disjunctive normal form be found. Thus the number of sentential variables in the combination is not a limiting factor for this method. In addition, a set of theorems will show that these simplified forms may be chosen to be minimal.

The given logical combinations will be assumed to be in disjunctive form. (From experience it appears that most logical functions are originally conceived in this form.) Advantage will be taken of special properties of the logical functions encountered in designing computing machines.

I. Methods of Changing from Disjunctive to Conjunctive Form

The sentential combinations considered in computing machine design are dictated by the over-all machine logic. This machine logic may show the impossibility of the occurrence of certain combinations of truth values of the sentential variables. Such combinations of sentential variables will be called 'undefined'. These combinations of truth values may be assigned any truth value without disturbing the over-all computer logic, (as long as the machine is working correctly). By choosing these truth values properly, simplifications may be made in the conjunctive form of expressions.

For example, let $f = (C_1 . A . B) \vee (C_2 . B . C . D)$, and let the truth value of any normal conjunction containing $C_1 . C_2$ be undefined. Using the usual laws of Boolean Algebra[1], factor B from f. Form the following disjunctions, $C_1 \vee C_2$, $C_1 \vee C$, $C_1 \vee D$,

(Received June 8, 1953)

$C_2 \vee A$, a simplified conjunctive form of f is

1) $f \text{ eq } B \cdot (C_1 \vee C_2) \cdot (C_1 \vee C) \cdot (C_1 \vee D) \cdot (C_2 \vee A)$.

This may be checked by preparing a truth table for f, using either a Harvard minimizing chart[5] or a chart of the form proposed by E. W. Veitch[7], and then simplifying the function.

Let a second sentential combination be $g \text{ eq } (C_1 \cdot A \cdot B) \vee (C_2 \cdot C) \vee (C_3 \cdot D \cdot E)$. As before, let any normal conjunction[1], containing $C_1 \cdot C_2$, $C_1 \cdot C_3$ or $C_2 \cdot C_3$ have undefined truth values. g may easily be written in the following simplified conjunctive forms.

2) $g \text{ eq } (C_1 \vee C_2 \vee C_3) \cdot (A \vee C_2 \vee C_3) \cdot (B \vee C_2 \vee C_3) \cdot (C \vee C_1 \vee C_3) \cdot$
$(D \vee C_2 \vee C_3) \cdot (E \vee C_2 \vee C_3)$.

3) $g \text{ eq } (C_1 \vee C_2 \vee C_3) \cdot (A \vee \bar{C}_1) \cdot (B \vee \bar{C}_1) \cdot (C \vee \bar{C}_2) \cdot (D \vee \bar{C}_3) \cdot$
$(E \vee \bar{C}_3)$.

1) and 2) follow from the application of some very simple rules, while 3) needs an additional rule.

If two sentences have the property that any normal conjunction containing their logical sum has an undefined truth value, then call these sentences command sentences. To differentiate when necessary, call all other sentences standard sentences.

The following rules will give a simplified conjunctive form from a disjunctive form.

Rules A:

A1) Factor the sentential combination into disjunctive form so that each disjunct contains one and only one command sentence and each command sentence appears in only one disjunct. (If two command sentences occur in a disjunct, define this disjunct for the function.)

A2) Factor all common factors from the expression.

A3) Make the first conjunct a disjunction of the command sentences.

A4) The remaining conjuncts in the conjunctive form will each contain exactly one standard sentence. Each conjunct that is formed is composed of a standard sentence and all command sentences not present with this standard sentence in the original disjunctions.

A5) The conjunction formed from the conjuncts of A3) and A4) is equivalent to the original function.

For example, let $f = C_1 . A . B \vee C_2 . A . D \vee C_3 . B . D$
By rules A, f eq $(C_1 \vee C_2 \vee C_3) . (A \vee C_3) . (B \vee C_2) . (D \vee C_1)$

Rule B:

Let $C_1, \ldots, C_n$ be n command sentences. Let $C_1, \ldots, C_n$ be divided into two disjoint sets $C_{i_1}, \ldots, C_{ik}$ and $C_{j_1}, \ldots, C_{jn}$. Let $A \vee C_{i_1} \vee \ldots \vee C_{ik}$ be a conjunct in a conjunctive form found by rules A. The conjunct $A \vee C_{i_1} \vee \ldots \vee C_{ik}$ may be replaced by the conjuncts formed from $(A \vee \overline{(C_{j_1} \vee \ldots \vee C_{jn})})$ to obtain an equivalent conjunctive form.

For example, applying rule B to 2) gives 3).

It should be stressed that the assignment of truth values to undefined sentential combinations made in rule A will change under the substitution in rule B.

Rules A may be generalized to apply to any sentential combination. Let there be no command sentences in the combination. In this case the sentences A and $\bar{A}$ (or any such pair) can act as command sentences and allow the application of rules A.

For example, let f eq $A . B \vee \bar{A} . C$
f eq $(A \vee \bar{A}) . (A \vee C) . (\bar{A} \vee B)$ eq $(A \vee C) . (\bar{A} \vee B)$.

More generally, let f eq $A . B . C \vee A . \bar{B} . D \vee \bar{A} . C . E \vee \bar{A} . \bar{E} . F$. In this case the form of rule A1 is obtained by successive reductions of the number of variables considered.

f eq $A . (B . C \vee \bar{B} . D) \vee \bar{A} (C . E \vee \bar{E} . F)$ eq $[A . (B \vee D) . (\bar{B} \vee C)] \vee [\bar{A} . (E \vee F) . (C \vee \bar{E})]$ eq $(A \vee E \vee F) . (A \vee C \vee \bar{E}) . (\bar{A} \vee B \vee D) . (\bar{A} \vee \bar{B} \vee C)$

The rest of this paper will contain the following material. First, proofs will be given of the validity of rules A and B. Secondly, there will be development of necessary and sufficient conditions for the forms found using rules A and B to be minimum forms. If the forms found using rules A and B are not minimal, extensions of these theorems will provide rules by which minimal forms can be found.

Many of the statements in this paper will be proved as follows.

Sentential combinations will be mapped onto a particular ring. Each element has degree two over a base field of characteristic two, and a simple minimum function. If two sentential combinations have the same algebraic map, then the two statements are equivalent, and conversely. Because of the simplicity of the minimum function and mod two algebra, algebraic manipulations furnish simple proofs of the equivalence of transforms.

The transformation rules are the following*.

$$
\begin{array}{lcl}
\text{False} & \longleftrightarrow & 0 \\
\text{True} & \longleftrightarrow & 1 \\
A & \longleftrightarrow & a \\
A + B & \longleftrightarrow & a + b \\
\underline{A} \,.\, B & \longleftrightarrow & a \times b \\
\bar{A} & \longleftrightarrow & a + 1 \\
A \vee B & \longleftrightarrow & a + b + a \times b
\end{array}
$$

Since $A \,.\, \bar{A} = F$, the minimum function for any element in the Boolean ring is given by: $a \times (a + 1) = 0$ or $a^2 = a$ over a base field of characteristic two. The usual associative and commutative laws can be proved to hold in the ring. In addition, the distributive law: $a \times (b + c) = (a \times b) + (a \times c)$ holds.

In particular, if C_1, C_2 are command sentences, then C_1, C_2 can never occur at the same time, so $C_1 \,.\, C_2$ can never occur, and $C_1 \,.\, C_2$ may be defined to be false. So if $C_1 \longleftrightarrow c_1$, $C_2 \longleftrightarrow c_2$, then $c_1 \,.\, c_2 = 0$.

The validity of rules A will now be shown.

Sentential combinations will be defined to be independent if the fixing of the truth value of one combination does not always imply the fixing of the truth value of the other.

Let N be the set of integers from 1 to n inclusive, and M be the similar set from 1 to m inclusive.

Lemma 1.

Let C_1, C_2 be command sentences. Let the elements A_i, i in N, B_j, j in M, be distinct and independent of C_1 and C_2. Then

4) $(C_1 \,.\, A_1 \,.\, \ldots \, A_n) \vee (C_2 \,.\, B_1 \,.\, \ldots \, B_m)$ eq $(C_1 \vee C_2) \,.$
$(C_1 \vee B_1) \ldots (C_1 \vee B_m) \,.\, (C_2 \vee A_1) \ldots (C_2 \vee A_n)$.

This lemma will be proved by transforming both the logical statements into algebraic statements. Let $n = m = 1$. 4) becomes $(C_1 \,.\, A_1) \vee (C_2 \,.\, B_1)$ eq $(C_1 \vee C_2) \,.\, (C_1 \vee B_1) \,.\, (C_2 \vee A_1)$. Now, $(C_1 \,.\, A_1) \vee (C_2 \,.\, B_1) \longleftrightarrow c_1 a_1 + c_2 b_1$ since from previous remarks $c_1 c_2 a_1 b_1$ may be defined to be zero.

$(C_1 \vee C_2) \,.\, (C_1 \vee B_1) \,.\, (C_2 \vee A_1) \longleftrightarrow (c_1 + c_2)\,(c_1 + b_1 + c_1\, b_1)$
$(c_2 + a_1 + c_2\, a_1) \longleftrightarrow [(c_1 + c_2)(c_1 + b_1 + c_1 b_1)].$
$[(c_1 + c_2)(c_2 + a_1 + c_2 a_1)] \longleftrightarrow [(c_1 + c_1 \cancel{b_1} + c_1 \cancel{b_1} + c_2 b_1].$
$[c_1\, a_1 + c_2 + \cancel{c_2} a_1 + \cancel{c_2}\, a_1] \longleftrightarrow c_1\, a_1 + c_2 b_1$

In general, clearly:

$$(C_1 \,.\, A_1 \,\ldots\, .\, A_n) \vee (C_2 \,.\, B_1 \,.\, \ldots \,.\, B_m) \longleftrightarrow c_1 \,.\, a_1 \ldots a_n + c_2\, b_1 \,..\, b_m$$

By induction, using the fact that $(C_1 \vee C_2)^n$ eq $C_1 \vee C_2$,

$$(C_1 \vee C_2)\,(C_1 \vee B_1) \ldots (C_1 \vee B_m) \longleftrightarrow c_1 + c_2\, b_1\, b_2 \ldots b_m$$
$$(C_1 \vee C_2)\,(C_2 \vee A_1) \ldots (C_2 \vee A_n) \longleftrightarrow c_2 + c_1\, a_1\, a_2 \ldots a_n$$
$$(C_1 \vee C_2)\,(C_1 \vee B_1) \ldots (C_1 \vee B_m) \,.\, (C_2 \vee A_1) \ldots (C_2 \vee A_n) \longleftrightarrow c_1\, a_1 \ldots a_n + c_2\, b_1 \ldots b_m$$

Use the usual notation' of $C_1 \vee C_2 \vee C_3$ eq $\prod_{j=1}^{3} C_j$ (logical product) and $C_1 \,.\, C_2 \,.\, C_3$ eq $\sum_{j=1}^{3} C_j$ (logical sum).

Lemma 2.

Let C_j, j in M, be command sentences, and let $A_{ij} \neq A_{ik}$, $j \neq k$ and all A_{ij} be independent of any C_j. Then

$$5)\quad (C_1 \vee C_2 \vee \ldots C_m) \,.\, \sum_{i=1}^{m} \left\{ \sum_{j=1}^{ni} A_{ij} \vee \prod_{k=1,\,..\,m}^{k\neq i} C_k \right\}$$

$$= \prod_{i=1}^{m} \left\{ C_i \,.\, \sum_{j=1}^{mi} A_{ij} \right\}$$

The proof of lemma 2 is similar to the proof of lemma 1 - in fact, lemma 1 is the verification step of the induction.

Lemmas 1 and 2 completely justify rules A in the case that the standard sentences are all distinct and independent. Usually, however, they are not distinct and independent, and in this case several simplifications are possible.

New notation will facilitate further progress. Let S_1, S_2 be two sets of integers.

The logical product $S_1 \vee S_2$ of these sets is simply the set of all integers in either set, while the logical sum, $S_1 \,.\, S_2$ is the set of all integers in both sets.

Lemma 3.

Let C_j, j in N, be command sentences, A be independent of the C_j. Let S_1, S_2 be two sets of integers contained in N. Then,

$$6)\quad \left(A \vee \prod_{j \notin S_1}^{j \text{ in } N} C_j\right) \cdot \left(A \vee \prod_{j \notin S_2}^{j \text{ in } N} C_j\right) \text{ eq } A \vee \prod_{j \notin S_1 \vee S_2}^{j \text{ in } N} C_j$$

The proof follows immediately from the mapping, the distributive law and the properties of the maps of the command variables.

The complete proof of rules A now follows easily after proper notation has been devised. Let S_1 be the set of all A_{ij}, $1 \leq j \leq n_i$, i in M. Let S_2 be the set of all distinct A_{ij}. Write

$$S_2 = \{B_k,\ 1 \leq k \leq R\}$$

where

$$R \leq \sum_{i=1}^{m} ni \ .\ B_k = A_{i_1 j_1} = A_{i_2 j_2} = \ldots = A_{ij}.$$

Define T_k to be the set of integers $\{i_1, i_2, \ldots, i_v\}$.

Theorem I.

Let C_i, i in M be a set of command sentences, and consider the sentential combination $\prod_{i=1}^{m} C_i \cdot \sum_{j=1}^{ni} A_{i_j}$. Form the sets $S_2 = \{B_k\}$, $T_k = \{i_1, i_2, \ldots, i_v\}$, $1 \leq k \leq R$ using the definitions in the last paragraph.

$$7)\quad \prod_{i=1}^{m} C_i \sum_{j=1}^{n_i} A_{ij} \text{ eq } (C_1 \vee \ldots \vee C_m) \cdot \left[\prod_{K=1}^{M} \left\{B_k \vee \prod_{i \notin TK}^{i \text{ in } M} C_i\right\}\right]$$

Proof: Apply 6) to 5) to get 7).

This completes the proof of rules A.

To prove rule B the following lemma is essential.

Lemma 4:

Let p, q, r, s be sentences, and any normal conjunction containing p . q have an undefined truth value. Then $\overline{p} \vee r$ eq $\overline{p} \vee r \vee q \cdot s$

Proof: $\overline{p} \vee r$ eq $\overline{p} \vee r \vee p \cdot q \cdot s$ eq $\overline{p} \vee r \vee q \cdot s$ where $p \cdot q \cdot s\ (r \vee \overline{r})$ are defined to be false normal conjunctions.

Consider the sentential combination $f = \prod_{i=1}^{m} C_i \cdot \sum_{j=1}^{n_i} A_{ij}$.

$f = \prod_{i=1}^{m} C_i \cdot \sum_{j=1}^{n_i} (\overline{C}_i \vee A_{ij})$. In this case all the terms $\overline{C}_i \vee A_{ij}$ are distinct and independent. By 5),

$$8)\quad f = (C_1 \vee C_2 \vee \ldots \vee C_m) \cdot \sum_{i=1}^{m} \left\{ \sum_{j=1}^{n_i} (\overline{C}_i \vee A_{ij}) \vee \prod_{\substack{k \text{ in } M \\ k \neq i}} C_k \right\}$$

$$= (C_1 \vee \ldots \vee C_m) \cdot \sum_{i=1}^{m} \left[\sum_{j=1}^{n_i} (\overline{C}_i \vee A_{ij}) \right]$$

using lemma 4.

Now combine the terms containing $B_k = A_{i_1 j_1} = \ldots = A_{i_k j_k}$ as defined earlier. These terms are

$$9)\quad (B_k \vee \overline{C}_{i_1}) \cdot (B_k \vee \overline{C}_{i_2}) \ldots (B_k \vee \overline{C}_{i_k}) = \left[B_k \vee \overline{(C_{i_1} \vee C_{i_2} \vee \ldots \vee C_{i_k}} \right.$$

The set $\{ i_1, \ldots, i_k \}$ is exactly the set T_k defined earlier.

Clearly these alterations may be made as is convenient, and rule B is proved.

Theorem 2. Using the hypotheses and definitions of Theorem 1,

$$10)\quad \prod_{i=1}^{m} C_i \cdot \sum_{j=1}^{n_i} A_{ij} \ \text{eq}\ (C_1 \vee \ldots \vee C_m) \cdot \left[\sum_{k=1}^{R} B_k \vee \overline{(\prod_{j \in T_k} C_j)} \right]$$

The conjuncts $B_k \vee \prod_{\substack{j=1 \ldots m \\ j \notin T_k}} C_j$ and $B_k \vee \overline{(\prod_{j \in T_k} C_j)}$ may be freely interchanged.

While these rules are easy to apply, they do not always result in so-called minimum forms. However, additional rules will enable these expressions to be changed to minimum forms.

II. Minimal Forms

In order to avoid questions of circuit design, a truth function f in conjunctive form will be defined to be reducible if it is possible to decrease,

(a) the number of disjuncts forming the conjunction without increasing the number of terms per disjunct, or

(b) to decrease the number of terms in a disjunct without increasing the number of disjuncts forming the conjunction.

Any truth function f in conjunctive form which is not reducible will be defined to be in minimum conjunctive form. This definition of minimum form does not depend upon the circuit analogue of the truth function; because, as engineering requirements change, so does a definition of minimum form based on circuit analogues.

However, if the minimum forms corresponding to a truth function f are known, it is easier to choose the most advantageous form for circuit design.

Theorems will be given which will show methods for reducing conjunctive forms, and tests which will indicate when these forms can no longer be reduced.

The first type of function to be considered will be:

$G = C_1 . f_1 \dots f_n \vee C_2 . g_1 \dots g_m$; C_1, C_2, are command sentences.

Conjunctive forms of G are as follows.

11) $G = (C_1 \vee C_2) . (g_1 \vee C_1) \dots (g_m \vee C_1) . (f_1 \vee C_2) \dots (f_n \vee C_2)$

$G = (C_1 \vee C_2) . (g_1 \vee \overline{C}_2) \dots (g_m \vee \overline{C}_2) . (f_1 \vee \overline{C}_1) \dots (f_n \vee \overline{C}_1)$

Some examples will quickly show that the forms 11) may not be minimum. Let, $f_1 = A$, $g_1 = \overline{A}$. Then, $G = (C_1 \vee C_2) . (A \vee C_1) \cdot (A \vee C_2)$ by 11). However, if A, $\overline{A}$ act as command pulses, $G = (A \vee \overline{A}) . (A \vee C_2) \cdot (\overline{A} \vee C_1) = (A \vee C_2) \cdot (\overline{A} \vee C_1)$.

For another example, let $f_1 = A$, $g_1 = A \vee B$. by 11), $G = (C_1 \vee C_2) . (A \vee C_2) \cdot (A \vee B \vee C_1)$. However, it is easily seen that $G = (C_1 \vee C_2) . (A \vee C_2) . (A \vee B)$ is a reduced form.

Let $f = A . B . C \vee \overline{A} . \overline{B} . \overline{C}$. At first glance it will seem that $f = A \vee \overline{B} . A \vee \overline{C} . \overline{A} \vee B . \overline{A} \vee C$. However, $B . C$ and $\overline{B} . \overline{C}$ have a common factor, namely $B \vee \overline{C}$, since $(B \vee \overline{C}) . C = B . C$ and $(B \vee \overline{C}) . \overline{B} = \overline{B} . \overline{C}$. Thus $f = (B \vee \overline{C}) . (A . C \vee \overline{A} . \overline{B}) = B \vee \overline{C} . A \vee \overline{B} . \overline{A} \vee C$, in its well-known minimum form, as shown in general in (5). This is a particular example of the following theorem.

Theorem 3.

If $f_i . g_i$, $i = 1, .., n$, are such that any conjunct containing them is either false or has an undefined truth value, then

12) $f_1 \cdot f_2 \cdot \cdots \cdot f_n = f_1 \cdot (f_2 \vee g_1) \cdot (f_3 \vee g_2) \cdot \cdots \cdot (f_n \vee g_{n-1})$

$g_1 \cdot g_2 \cdot \cdots \cdot g_n = g_n \cdot (f_2 \vee g_1) \cdot (f_3 \vee g_2) \cdot \cdots \cdot (f_n \vee g_{n-1})$.

The proof by expression is obvious.

This theorem frequently is useful in finding the factors and the factored form of $F = C_1\, f_1 \ldots f_n \vee C_2\, g_1 \ldots g_m = p(C_1\, f_1' \ldots f_j' \vee C_2\, g_1' \ldots g_k')$.

Let the combination of truth values $p_1 = p_2 = \ldots = p_k$ = True and $p_{k+1} = \ldots = p_m$ = False never occur. A notation to indicate this will be

13) $p_1 \cdot p_2 \cdot \cdots \cdot p_k \cdot \bar{p}_{k+1} \cdot \cdots \cdot \bar{p}_n = 0$.

Impossible combinations may also be used in another way, as shown by the following example.

Let $B \cdot C = 0$, and $G = A \cdot \bar{B} \cdot \bar{C} \vee \bar{A} \cdot C$. Since $B \cdot C = 0$, $\bar{A} \cdot C = \bar{A} \cdot \bar{B} \cdot C \vee \bar{A} \cdot B \cdot C = \bar{A} \cdot \bar{B} \cdot C$, so $G = \bar{B} \cdot (A \cdot \bar{C} \vee \bar{A} \cdot C) = \bar{B} \cdot (\bar{A} \vee \bar{C})$ in minimum conjunctive form.

Now consider the case when $f = f_1 \cdot \cdots \cdot f_n$ and $g = g_1 \cdot \cdots \cdot g_m$ do not have a common factor. The assumption the $C_1\, f_1 \ldots f_n \neq 0$, $C_2\, g_1 \ldots g_m \neq 0$ will also be made, since either of these combinations being impossible gives a trivial reduction of G.

To find conditions under which 11) is a minimal form, first consider the conjuncts. The problem is to find a disjunction containing g_i which is, first, contained in the conjunctive form of G and, secondly, has the fewest number of terms.

Consider the disjunction consisting of g_i itself. If this were a conjunct in the conjunctive form of G, then g_i eq F would imply G eq F. Thus, $g_i \vee \bar{G}$ eq T, or

14) $G \cdot \bar{g}_i$ eq $(C_1\, f_1 \ldots f_n \vee C_2\, g_1 \ldots g_m) \cdot \bar{g}_i$ eq $C_1\, f_1 \ldots f_n \cdot \bar{g}_i$ eq F.

Conversely, if $C \cdot f \cdot \bar{g}_i$ eq F or is undefined, then G eq $G(g_i \vee \bar{g}_i)$ eq $G \cdot g_i$ eq $g_i \cdot (C_1\, f \vee C_2\, g_1 \ldots g_{i-1} \cdot g_{i+1} \ldots g_m)$ and g_i is a disjunction in a conjunctive form of G. Similar statements hold for f_i.

In the following statements use will be made of the well-known theorem that G is in a minimum conjunctive form if and only if $\bar{G}$ is in a minimum disjunctive form.

$$15)\quad \overline{G} = \overline{C}_1 . \overline{C}_2 \vee \prod_{i=1}^{m} \overline{C}_1 . \overline{g}_i \vee \prod_{j=1}^{n} \overline{C}_2 . \overline{f}_i \vee \prod_{i,j} \overline{f}_i . \overline{g}_i$$

Set $\overline{f}_j . \overline{g}_i$ eq T. Then if $\overline{G} = \overline{f}_j . \overline{g}_i \vee g$, after substitution

$$16)\quad g \text{ eq } \overline{C}_1 \vee \overline{C}_2 \vee \prod_{j \neq i} \overline{f}_j \vee \prod_{i \neq j} g_i \text{ eq T, since } C_1 . C_2 \text{ eq T,}$$

is impossible. Thus all disjuncts of the form $\overline{f}_j . \overline{g}_i$ may be discarded[3].

Each disjunct in an irredundant equivalent of G may be expanded into a set of normal disjuncts of G.[1] The totality of these normal disjuncts may be used to form the distinguished disjunctive normal form of G. This form defines G completely, and has been used as a basis for the theorems of sentential calculus. These disjuncts have also been the basis for most minimization methods.[3,5,7]

Consider the set of normal disjuncts formed from the disjuncts in an irredundant form of G. Some normal disjuncts will appear in the expansion of several irredundant disjuncts, while others will appear in the expansion of only one irredundant disjunct. Normal disjuncts of the second kind will be said to correspond uniquely to an irredundant disjunct.

Lemma 4.

Let $\overline{G} = \prod_{i=1}^{m} \overline{C}_1 . \overline{g}_i \vee \prod_{j=1}^{n} \overline{C}_2 . \overline{f}_j \vee \overline{C}_1 . \overline{C}_2$; C_1, C_2 be command sentences. The disjuncts corresponding uniquely to the irredundant disjuncts of $\overline{G}$ are listed below.

Component	Corresponding Defining Disjunct
$\overline{C}_1 . \overline{g}_i$	$(\overline{C}_1 . \overline{g}_i) . \sum_{k \neq i}^{k \text{ in } M} (C_1 \vee g_k) . \sum^{j \text{ in } N} (C_2 \vee f_j) . (C_1 \vee C_2)$
$\overline{C}_2 . \overline{f}_j$	$(\overline{C}_2 . \overline{f}_j) \sum^{i \text{ in } M} (C_1 \vee g_k) \sum_{k \neq j}^{k \text{ in } N} (C_2 \vee T_k) . C_1 \vee C_2$
$\overline{C}_1 . \overline{C}_2$	$(\overline{C}_1 . \overline{C}_2) . \sum^{i \text{ in } M} (C_1 \vee g_i) . \sum^{j \text{ in } N} (C_2 \vee f_j)$

Proof:

$$(\overline{C}_1 . \overline{g}_i) . \sum_{k \neq i}^{k \text{ in } M} (C_1 \vee g_k) . \sum^{j \text{ in } N} (C_2 \vee f_j) . (C_1 \vee C_2)$$

$$17)\quad = \overline{C}_1 \cdot \overline{g}_i \cdot C_2 \cdot (C_1 \vee \prod_{k \neq i}^{k \text{ in } M} g_k) \cdot (C_2 \vee \prod^{j \text{ in } N} f_j)$$

$$= \overline{C}_1 \cdot C_2 \cdot g_1 \cdot \cdots g_{i-1} \cdot \overline{g}_i \cdot g_{j+1} \cdot \cdots\cdots g_m$$

$$= C_2 \cdot g_1 \cdots g_{i-1} \cdot \overline{g}_i \cdot g_{i+1} \cdots g_m \cdot$$

Now, $\overline{C}_1 \cdot \overline{g}_i$ in normal disjunctive form may be written as $\overline{C}_1 \cdot \overline{g}_i \cdot N$ $(C_2 \ g_k T_j)$, k in M, $k \neq i$, j in N.

All disjuncts containing $\overline{g}_k$, $k \neq i$ are also in the normal form corresponding to $\overline{C}_1 \cdot \overline{g}_k$. Thus only affirmative g_k, $k \neq i$, are left.

Similarly only disjuncts containing C_2 may be left. Since the f_j are unrestricted the normal disjuncts implied uniquely by $\overline{C}_1 \cdot \overline{g}_i$ are given by the stated expression.

The proofs for $\overline{C}_2 \cdot \overline{f}_j$, $\overline{C}_1 \cdot \overline{C}_2$ are similar.

$$18)\quad (\overline{C}_2 \cdot \overline{f}_j) \cdot \sum^{i=1,..m} (C_1 \vee g_i) \cdot \sum_{k \neq j}^{k=1,..n} (C_2 \vee f_k) \cdot (C_1 \vee C_2)$$

$$= C_1 \cdot \overline{C}_2 \cdot f_1 \cdots f_{j-1} \cdot \overline{f}_j \cdot f_{j+1} \cdot \cdots f_n$$

$$= C_1 \cdot f_1 \cdots f_{j-1} \cdot \overline{f}_j \cdot f_{j+1} \cdot \cdots f_n$$

$$19)\quad (\overline{C}_1 \cdot \overline{C}_2) \sum^{i=1,..m} (C_1 \vee g_i) \sum^{j=1,..n} (C_2 \vee f_j)$$

$$= \overline{C}_1 \cdot \overline{C}_2 \cdot g_1 \cdots g_n \cdot f_1 \cdots f_n$$

Using these lemmas the following main theorem and its attendant corollaries may be proved.

Theorem 4.

Let $G = C_1 f_1 \ldots f_n \vee C_2 g_1 \ldots g_m$; C_1, C_2 be command sentences, let no expressions of the form 14), 17), 18) or 19) have undefined truth values or be false, $C_1 \cdot f \neq 0$, $C_2 \cdot g \neq 0$, and let f, g have no common factors. Then minimum conjunctive forms for G are

$$11)\quad G = (C_1 \vee C_2) \cdot (C_1 \vee g_1) \ldots (C_1 \vee g_m) \cdot (C_2 \vee f_1) \ldots (C_2 \vee f_n)$$

$$G = (C_1 \vee C_2) \cdot (\overline{C}_1 \vee f_1) \ldots (\overline{C}_1 \vee f_n) \cdot (\overline{C}_2 \vee g_1) \ldots (\overline{C}_2 \vee g_m)$$

Note that $C_1 \vee g_i$ and $\overline{C}_2 \vee g_i$ and $C_2 \vee f_j$ and $\overline{C}_1 \vee f_j$ may be freely interchanged.

C_1 does not appear as a conjunct, since if it did, then $\overline{C}_1 \rightarrow \overline{G}$, or $\overline{C}_1 \cdot G$ eq F or $C_2 \cdot g$ eq F, which is impossible.

Assume that some disjunct, say $C_1 \vee g_i$, does not appear (this statement will be proved false). If $C_2, g_1, \ldots g_m$ are all true, then G is true. However, C_1 is false since C_1, C_2 are command variables. Now let g_i be false, and C_2 and all other g_j be true. The assumed conjunctive form of G is unchanged (g_i does not appear), so if this conjunctive form of G exists then G is true for this set of variables. However, since C_1 is false, G is seen to be false from the disjunctive form unless the truth combination G_i false, C_2 and all other g_j true is an impossible combination. However, $C_2 \, g_1 \cdot \ldots g_{i-1} \, \overline{g}_i \, g_{i+1} \ldots g_m \neq 0$ by the assumptions of the theorem and the assumption that $C_1 \vee g_i$ does not appear has lead to a contradiction. Thus each conjunct appears. The same proof holds for $C_2 \vee f_j$.

Using Lemma 4 it is clear that to $\overline{C}_1 \cdot \overline{C}_2$ there corresponds uniquely a normal disjunct which does not correspond uniquely to any other irredundant two term disjunct.

Thus, $\overline{C}_1 \cdot \overline{C}_2$ lies in the core of F,[3] and, in this case, since 19) $\neq 0$, $C_1 \vee C_2$ always is in the minimum conjunctive form of F.

This completes the proof of the theorem.

The hypotheses of the theorem will now be considered in detail to see if their weakening will simplify the conjunctive forms of G.

Corollary 1:

Let $G = C_1 \cdot f_1 \ldots . f_n \vee C_2 \cdot g_1 \ldots g_m$; C_1, C_2 be command sentences, let so expressions of the form 14), 17) or 18) have undefined truth values let $C_1 \, f \neq 0$, $C_2 \, g \neq 0$ and let f and g have no common factors. The conjunct $C_1 \vee C_2$ can be dropped from the form 11) if and only if $\overline{C}_1 \cdot \overline{C}_2 \cdot f_1 \ldots f_n \cdot g_1 \ldots g_m$ is false or has an undefined truth value. In this case $C_1 \vee g_i$ and $\overline{C}_2 \vee g_i$ and $C_2 \vee f_j$ and $\overline{C}_1 \vee f_j$ may not be freely interchanged.

If $\overline{C}_1 \cdot \overline{C}_2 \cdot f \cdot g$ has an undefined truth value, then by the proof above $C_1 \vee C_2$ is not needed.

If $C_1 = \overline{C}_2$, $C_1 \vee C_2$ eq $\overline{C}_2 \vee C_2$ eq T is superfluous. If $f_i = \overline{g}_j$ use f_i, g_j as command sentences. The converse is clear from the proof of the theorem.

If G is written in the second conjunctive form of 11), then the disjuncts corresponding uniquely to $C_1 \vee C_2$ are given by $\overline{C}_1 \cdot \overline{C}_2$.

The corresponding defining conjunctions 17) and 18) are unchanged. However, if $C_1 \vee C_2$ can be dropped from the first form of 11), then 17) becomes

17*) $\overline{C}_1 \cdot g_1 \cdot\cdot g_{i-1} \cdot \overline{g}_i \cdot g_{i+1} \cdot\cdot g_m (C_2 \vee f)$

18) becomes

18*) $\overline{C}_2 \cdot f_1 \ldots f_{j-1} \cdot \overline{f}_j \cdot f_{j+1} \ldots f_n (C_1 \vee g_1 \ldots g_m)$

Thus the interchange of $\overline{C}_1 \vee f_j$ and $C_2 \vee f_j$ is not always possible.

It might well be noted that if $f_j \cdot g_i = 0$, then 19) = 0 and the corollary applies. This condition is frequently met in practice.

Corollary 2

Let $G = C_1 \cdot f_1 \ldots f_n \vee C_2 \cdot g_1 \ldots g_m$; C_1, C_2 be command sentences. If $C_2 \cdot g_1 \cdot\cdot g_{i-1} \cdot \overline{g}_i \cdot g_{i+1} \cdot\cdot g_m$ is false or has an undefined truth value, then g_i may be dropped from the disjunctive form and $C_1 \vee g_i$ ($\overline{C}_2 \vee g_i$) from the conjunctive form. Similarly, if $C_1 \cdot f_1 \ldots f_{j-1} \cdot \overline{f}_j \cdot f_{j+1} \ldots f_n = 0$, then f_j may be dropped from the disjunctive form and $C_2 \vee f_j$ ($\overline{C}_1 \vee f_j$) from the conjunctive form.

Define $C_2 \cdot g_1 \cdot\cdot g_{i-1} \cdot \overline{g}_i \cdot g_{i+1} \cdot\cdot g_m$ eq F.
G eq G $\vee C_2 \cdot g_1 \cdot\cdot g_{i-1} \cdot \overline{g}_i \cdot g_{i+1} \cdot\cdot g_m$
eq $C_1 f \vee C_2 g_1 \cdot\cdot g_{i-1} \cdot g_{i+1} \cdots g_m$.
The rest is obvious.

Corollary 3

Let $G = C_1 f_1 \ldots f_n \vee C_2 g_1 \ldots g$; and C_1, C_2 be command sentences; f, g have no common factors.

g_i (f_j) are conjuncts in a minimum conjunctive form of G if and only if $C_1 \cdot f \cdot \overline{g}_i = 0$ ($C_2 \cdot g \cdot \overline{f}_j = 0$).

This was essentially proved earlier.

Corollary 2 and Corollary 3 may clearly be applied as many times as hypotheses are fulfilled, and each application yields a reduced form.

Let K be a logical statement concerning C_1, C_2, f_1, g_1, .. g_m. Let K be such that K = 0 does not imply, for some i, j, that 14) = 0.

or 17) = 0, or 18) = 0, or 19) = 0. Now K = 0 will not decrease the number of conjuncts in the conjunctive form 11) as long as f, g have no common factors. This is true by Lemma 4 since all disjuncts in the disjunctive normal form of $\overline{G}$ must correspond to some disjunct in the minimum form of $\overline{G}$, and by the lemma all terms in 11) correspond to necessary terms in $\overline{G}$. Neither will K = 0 decrease the number of terms in any disjunct, for in order for this to happen $C_1\ f\ .\ \overline{g_i}$ or $C_2\ g\ \overline{f_j}$ must be false or undefined for some i or j.

Theorem 5:

Let $G = C_1\ .\ f \vee C_2\ .\ g$; C_1, C_2 be command sentences and f, g have no common factors. Theorem 4 and its corollaries express all possible reductions in the conjunctive form of G.

The next case to be considered is that in which f and g have a common factor. In this case factor f and g into $f = Sf^*$ and $g = Sg^*$ and assume that the number of conjuncts in f^* and g^* is less than the number of conjuncts in f and g. It will be assumed that a factoring in which F reduces to the previous case does not occur, in other words $\overline{S}\ .\ (C_1\ f^* \vee C_2\ g^*) \not\equiv 0$.

Proofs similar to those carried out previously are valid, and the following theorem may be obtained, where S is any logical expression in conjunctive form.

Theorem 6:

Let $F = C_1\ f^* \vee C_2\ g^* = S\ .\ (C_1\ f \vee C_2\ g) = S\ .\ (C_1\ f_1 \ldots f_n \vee C_2\ g_1 \ldots g_m)$; C_1, C_2 be command sentences; $C_1\ f_1 \ldots f_n \not\equiv 0$, $C_2\ g_1 \ldots g_m \not\equiv 0$, $\overline{S}\ .\ (C_1\ f \vee C_2\ g) \not\equiv 0$, f and g have no common factors which decrease the length of $f_1 \ldots f_n$ and $g_1 \ldots g_m$. In conjunctive form

$$F = S\ .\ C_1 \vee C_2\ .\ C_1 \vee g_1 \ldots C_1 \vee g_m\ .\ C_2 \vee f_1 \ldots C_2 \vee f_n$$

Superfluous Disjunct	Necessary and Sufficient Condition for Reduction
$C_1 \vee C_2$	$S\ .\ \overline{C}_1\ \overline{C}_2\ f\ .\ g = 0$
$C_1 \vee g_i$	$S\ .\ C_2\ g_1 \ldots g_{i-1}\ \overline{g_i}\ g_{i+1} \ldots g_m = 0$
$C_2 \vee f_j$	$S\ .\ C_1\ f_1 \ldots f_{j-1}\ \overline{f_j}\ f_{j+1} \ldots f = 0$
$C_1 \vee g_i$ reduced to g_i	$S\ .\ C_1\ f\ \overline{g_i} = 0$
$C_2 \vee f_j$ reduced to f_j	$S\ .\ C_2\ g\ \overline{f_j} = 0$

Similar statements can be written replacing $C_1 \vee g_i$ by $\overline{C}_2 \vee g_i$

and $C_2 \vee f_j$ with $\overline{C}_1 \vee f_j$ if proper care is taken with the replacement of $C_1 \vee C_2$.

Theorem 3 is useful in finding factors S.

Remember that in practical applications if, for example, $S = p \vee q$, $m = n = 3$, then $p \cdot g_1 = 0$, $q \cdot g_2 = 0$ would imply that $C_1 \vee g_3$ was superfluous.

Now consider the general case in which

$$20)\quad G = \prod_{i=1}^{n} \left(\sum_{j=1}^{n_i} C_i f_{i_j} \right) = \prod_{i=1}^{n} C_i g_i \text{ where } C_i \cdot C_j = 0 \text{ for } i \neq j.$$

A minimum conjunctive form for G may be built step by step by using the previous theorems. First, a minimum conjunctive form for $C_1 \cdot g_1 \vee C_2 \cdot g_2 = (C_1 \vee C_2)\, h_2 = (C_1 \vee C_2)\, (\overline{C}_1 \vee f_{11}) \ldots (\overline{C}_1 \vee f_{1n_1})$. $(\overline{C}_2 \vee f_{21}) \ldots (\overline{C}_2 \vee f_{2n_2})$ may be found, using Theorem 4 and it's corollaries or Theorem 6. Now, a conjunctive form may be found for $C_1\, g_1 \vee C_2\, g_2 \vee C_3\, g_3 = (C_1 \vee C_2)\, h_2 \vee C_3\, g_3 = (C_1 \vee C_2 \vee C_3)\, h_3$ using Theorem 4 and its corollaries or Theorem 6. $h_3 = (\overline{C}_1 \vee f_{11}) \ldots$ $(\overline{C}_1 \vee f_{1n_1}) . (\overline{C}_2 \vee f_{21}) \ldots (\overline{C}_2 \vee f_{2n_2}) . (\overline{C}_3 \vee f_{31}) \ldots (\overline{C}_3 \vee f_{3n_3})$. In forming h_3 use was made of the identity $\overline{C}_1 \vee f_{11} \vee \overline{C_1 \vee C_2}$ eq $\overline{C}_1 \vee f_{11}$ and similar identities.

Now consider the conditions for dropping a conjunct. $C_1 \vee C_2 \vee C_3$ may be dropped if $C_1 \vee C_2 \vee C_3$ eq T. This, of course, can hold only if $n = 3$. If $n \neq 3$, there is no reason to consider replacing $\overline{C}_i \vee f_{i_j}$ with f_{i_j} since the next step in the method would replace f_{i_j} by $f_{i_j} \vee (\overline{C_1 \vee C_2 \vee C_3})$ a more complicated set of conjuncts. The condition for dropping $\overline{C}_i \vee f_{i_j}$ is $C_i \cdot f_{i_1} \ldots f_{i_{j-1}} \cdot \overline{f}_{i_j} \cdot f_{i_{j+1}} \ldots f_{i_{ni}} = 0$. This condition depends only upon the original disjunct and thus may be applied at any time.

As an example consider $G = C_1\, f_1 \vee C_2\, f_2 \vee C_3\, f_3$ and see the condition which must be fulfilled if f_3 is to be a conjunct in the conjunctive form of G. By lemma 4 $\overline{C}_1 \vee f_1$ eq $\overline{C}_2 \vee C_1 \vee f_1$.

The condition that f_3 is a conjunct thus may be transformed as follows: $[(C_1 \vee C_2) . (\overline{C}_1 \vee f_1) . (\overline{C}_2 \vee f_2)] . \overline{f}_3$ eq $[(C_2 \vee C_1\, f_1)$.

$(C_1 \vee C_2 f_2)] \cdot \bar{f}_3$ eq $(C_1 f_1 \vee C_2 f_2) \cdot \bar{f}_3$.

By induction, f_{jk} becomes a conjunct in the conjunctive form of G defined by 20) if and only if

$$21) \quad \Big(\prod_{\substack{i \text{ in } N \\ i \neq j}} C_i g_i\Big) \cdot \bar{f}_{jk} = 0$$

Thus all conditions for reducing the conjunctive form of G are independent of the choice of order in which the command sentences are taken, and a unique reduced form is obtained.

Alternate forms may be obtained by the use of Theorem 2. In other words $(C_1 f g).(C_2 \vee g)$ may be replaced by

$$(g \vee \prod_{\substack{i \text{ in } N \\ i \neq 1,2}} C_i)$$

Now assume that g_i, $i = 1, \ldots n$, have a common factor p. Now, $p\, g_i = g_i$, $i = 1, \ldots n$, and $p\, g_i = p\, f'_{i1} \ldots f'_{im_i}$. $i = 1, \ldots n$. The sum $\sum_{i=1}^{m} m_i$ may be compared with $\sum_{i=1}^{n} n_i$ the number of original terms. If K is the number of conjuncts in p, and if $K + \sum_{i=1}^{m} m_i \leq \sum_{i=1}^{n} n_i$ factoring p from G will lead to a conjunctive form with fewer disjuncts of fewer terms per disjunct. Make the assumption that the g_i, $i = 1, \ldots, n$ have no common factor p, with k conjunctive terms, such that

$$k + \sum_{i=1}^{m} m_i < \sum_{i=1}^{n} n_i$$

In addition, assume that $C_i \cdot g_i \neq 0$, $i = 1, \ldots, n$. Now consider combining these statements into a theorem.

Theorem 7.

Let $G = \prod_{i=1}^{n} \big(\sum_{j=1}^{n_i} C_i f_{ij}\big) = \prod_{i=1}^{n_i} C_i g_i$, $C_i \cdot C_j = 0$, $i \neq j$, $C_i g_i \neq 0$, $i \neq 1 \ldots n$. Let g_i, $i = 1 \ldots n$, have no common factors p, with k conjunctive terms, $g_i = p \cdot g_i' = p\, f_{i1}' \ldots f_{in_i}'$, such that $k + \sum_{i=1}^{m} m_i \leq \sum_{i=1}^{n} n_i$.

In conjunctive form $G = C_1 \vee \ldots\ldots \vee C_n \cdot \sum_{i,j} \overline{C}_i \vee f_{ij}$ . . .

Certain reductions are as follows:

Superfluous Disjuncts	Necessary Condition
$C_1 \vee \ldots \vee C_n$	$\overline{C}_1 \ldots\ldots \overline{C}_n = 0$
$\overline{C}_1 \vee f_{i_j}$ replaced by f_{i_j}	$\left(\prod_{\substack{k \text{ in } N \\ k \neq i}} C_k g_k \right) \overline{f}_{i_j} = 0$
$\overline{C}_i \vee f_{i_j}$	$C_i f_{i_1} \ldots f_{ij-1} \overline{f}_{ij} f_{ij+1} \ldots f_{in_i} = 0$

Possible reductions may be obtained by applying Theorem 2.

The case in which $G = p \cdot \prod_{i=1}^{n} C_i \cdot g_i$ may be treated, as before, by simply changing the necessary conditions by forming the conjunction of p and the necessary condition expressed above.

BIBLIOGRAPHY

1. D. Hilbert and W. Ackerman, Principles of Mathematical Logic, Chelsea Publishing Co., N. Y., 1950.
2. R. C. Jeffrey, I. S. Reed, The Use of Boolean Algebra in Logical Design, Digital Computer Lab, M. I. T.
3. W. V. Quine, The Problem of Simplifying Truth Functions, American Mathematical Monthly, vol. LIX, No. 8, October 1952.
4. Claude E. Shannon, A Symbolic Analysis of Delay and Switching Circuits, Trans. Am. Inst. of Electronic Engineers, vol. 57, 1938.
5. Synthesis of Electronic Computing and Control Circuits, vol. 27, Annals of the Harvard Computation Laboratory.
6. E. C. Varnum, 'Relay Circuit Analysis', Machine Design Magazine, December 1949.
7. E. W. Veitch, A Chart Method for Simplifying Truth Functions, Association for Computing Machinery Meeting, Pittsburgh, Penn. May 2, 1952.

RAYTHEON MFG. CO.

WALTHAM, MASSACHUSETTS

NOTES ON DECISION ELEMENT SYSTEMS USING VARIOUS PRACTICAL TECHNIQUES

JOHN D. GOODELL

The logic of Decision Elements, an interpretation of the Calculus of Propositions, is applicable to all techniques for the construction of digital computing devices.[1] It has been demonstrated many times that complete systems may be constructed using only the element (or logical functor) S or the element D. From the logical point of view, and using some physical techniques, this is entirely valid. As a practical matter, there are a number of desirable techniques where it is necessary to have a minimum of two types of D.E.'s available as well as a clock possible generator (V). One such system uses Magnetic Decision Elements in which all of the necessary functions are incorporated in a practical 'building block' design for the construction of digital computers. In such a device four basic functions must be incorporated:

(a) Decision (b) Power Gain (c) Temporary Storage
(d) Pulse Shaping

In Magnetic Decision Elements (b) is accomplished by differential loading on the clock pulse versus the intelligence pulse, (c) is handled by storage in rectangular hysteresis loop materials, (d) from the quantifying characteristics of such magnetic materials when flux swings are between limits of saturation and (a) by diode mixing and/or magnetic cancellation effects.

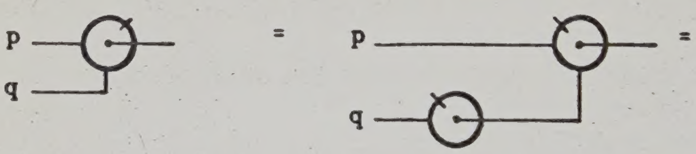
The input information is recognized by such magnetic materials as a pulse that changes the state of the flux from saturation in one direction to saturation in the other direction, and the output information is generated as a pulse that results from a flux swing caused when a clock-pulse signal applied to the magnetic material in the correct sense to return it to its normal or zero state. If no input pulse has been received, the clock pulse moves no flux and the output is 'no pulse', i.e., 0. It is evident that such a device cannot generate an output at the same time that it is receiving an input. Consequently, such Decision Elements must be divided

(Received June 28, 1953)

into two sets and operated from two sources of clock pulses staggered with respect to their time phasing. Thus, in any circuit of Magnetic Decision Elements, successive alternate elements are connected to opposite time phases of clock pulse. In practice these two clock pulse trains are derived from a center-tapped output transformer associated with a single clock-pulse generator. This means that if the primary oscillator source operates at, for example, 200 kilocycles, then the actual rate of information travel through a complex configuration of Magnetic Decision Elements will be at the rate of 100 kilocycles, although the actual pulse width is 5 microseconds.

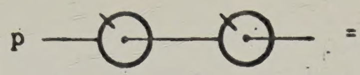
The information is 'stepped' along through the circuit from one set of elements to the other set of elements in an interlaced pattern. Five microseconds are consumed in applying input information to one set of elements, and at the same time five microseconds are consumed in reading information out of the other set. The total time from the beginning of the input pulse applied to any element to the termination of the output pulse is ten microseconds. In order to handle this conveniently in the Decision Element notation, one clock pulse train is designated α and the other β, and the corresponding symbols are inserted in the circle at the top.

Under the conditions described it is not possible to establish a complete system using only one type of element, for example S. Assume, for example, that it is necessary to synthesize the element T as follows:[2]



$$SpSq\Pi rr = SpNq = NLpq = NCqp = Tpq$$

Evidently the insertion of a single input S for negation on the input q will result in applying the inputs p and q at different times, and it is necessary to insert a suitable time delay in the line for p. In order to introduce a time delay that does not affect the intelligence content, Assertium for one argument is required. But this is constructed from two S elements in series![3]



$$SSp\Pi qq\Pi qq = SNpNp = A_1 p = \text{Asertium } p$$

Thus a new unwanted time delay is applied to the input p and problem leads to a contradiction.

In order to resolve this in a system based on S, the most convenient element to use in magnetic techniques is A. The decision required from A is readily obtained by an input diode mixer, and no magnetic cancellation effects are involved. Furthermore, A with one input functions as Assertium and constitutes a means of constructing delay lines of any required length by connecting single-input A elements in series without affecting the intelligence. Finally, A is the complement of S and thus corresponds to output negation of S. Thus many negations that would ordinarily be necessary in a system depending solely on S can be accomplished by substitution of A for S or S for A.

Practical devices of this kind have been constructed and are now being used in the design of digital computers. It is of passing interest that R and K have also been designed using similar magnetic structures in order to minimize the number of Decision Elements required in circuits where these functors are extensively used. There is a relatively minor advantage in having individual elements K and R available in a form that requires only one unit of operating time, since when these elements are synthesized, two or more units of time may be involved, and in some types of serial machine designs this factor represents a gain. However, in many machine designs the logic that develops out of an exclusively S/A system is such that the availability of K and R as separate elements is of minor import.

NOTES

1. There is no implication that the systems that have been described in previous papers for the logic of Decision Elements and the special notations that have been developed are necessarily optimum for handling the analysis of all practical techniques. In some cases, particularly where special devices of complex input/output logical characteristics happen to be desirable from the physical packaging point of view, other notations and concepts

may be more convenient. However, the Decision Element logic and notations are very basic in nature and any binary system may be interpreted and analyzed in these terms.

2 - 3. From the tables of definitions for an S/A system given on pages 97 and 98 of Vol.1, No. 2, The Journal of Computing Systems.